SIGNAL RECOVERY

Exercise book

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PROBLEM 1

June 25th, 2018 – Pb. 1

In this exercise, the recovery of a signal composed by three adjacent rectangles is discussed. Optimum filtering, variable-parameter filtering, and digital filtering are considered in presence of only white noise. Finally, 1/f noise also comes into play.

Consider the signal shown in Fig 1. The characteristics of the preamplifier used to read out the signal are specified below.

A) Evaluate the minimum measurable amplitude $V_{P,MIN}$ without using any additional filter. Then, describe and explain the ideal filter that makes it possible to measure the pulse amplitude V_P with the best possible Signal-to-Noise ratio and evaluate the minimum amplitude $V_{P,MIN}$ thus measurable.

B) Consider now to employ filters with variable parameters. Select a suitable practical filter, select its parameters to maximize the Signal-to-Noise ratio (S/N) and evaluate the minimum amplitude $V_{P,MIN}$ that can be measured in these conditions.

C) Consider now to follow a fully digital approach. Discuss the guidelines to select the sampling frequency and how this choice could have an impact on the Signal-to-Noise ratio. Choose a reasonable value for the sampling frequency and evaluate the minimum amplitude $V_{P,MIN}$ that can be measured in this case.

D) Consider now an additional 1/f noise component with a corner frequency $f_C = 5kHz$ affecting the preamplifier. Discuss the impact of this additional noise component on the final S/N and propose at least two different solutions to minimize this effect. Choose one of the two proposed solutions, provide quantitative data (e.g. filter parameters) and evaluate the minimum amplitude $V_{P,MIN}$ that can be measured in these conditions.

Pulse signal
V_P variable pulse amplitude
Preamplifier
$\sqrt{S_V} = 20nV/\sqrt{Hz}$ white noise spectral density (unilateral)
$f_{PA} = 50MHz$ upper band-limit

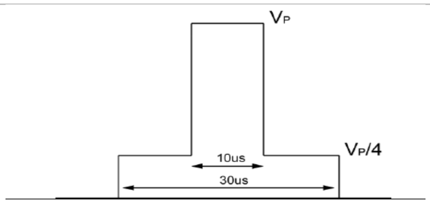


Fig.1

SOLUTION

In this problem, the amplitude V_P of a signal of known shape is to be measured. The signal shape can be described by the juxtaposition of three rectangles, each one of duration $T_P = 10\mu s$, as follows:

$$s(t) = \begin{cases} \dfrac{V_P}{4} & for\ 0 < t < T_P \\ V_P & for\ T_P < t < 2T_P \\ \dfrac{V_P}{4} & for\ 2T_P < t < 3T_P \end{cases}$$

The signal is read out by means of a preamplifier featuring a single pole $\mathrm{f_{PA}} = 50\mathrm{MHz}$; such preamp provides a low-pass filtering action on the input with $\tau = \frac{1}{2\pi \mathrm{f_{PA}}} = 3.18ns$. Since $\tau \ll T_P$, the shape of each rectangle is unchanged, and so is the overall waveform.

A) *Evaluate the minimum measurable amplitude $V_{P,MIN}$ without using any additional filter. Then, describe and explain the ideal filter that makes it possible to measure the pulse amplitude V_P with the best possible Signal-to-Noise ratio and evaluate the minimum amplitude $V_{P,MIN}$ thus measurable.*

First of all, the SNR without any additional filter must be computed. As explained above, the preamplifier does not have any impact on the signal, i.e. the signal shape at the output of the preamplifier coincides with the signal fed to the preamp input, which is shown in Fig. 1 and has been mathematically described above. The preamplifier itself adds white noise to the signal of interest, with unilateral spectral density $\sqrt{S_V} = 20nV/\sqrt{Hz}$. Such noise contribution is upper limited by the pole of the preamp, acting as a low pass filter, and possibly lower limited by a zero setting of the instrument that is used to read-out the signal (e.g. an oscilloscope). It is worth noting that an estimation of the zero setting is not necessary here. Indeed, its contribution to noise filtering, that is inversely proportional to the time distance between the initial acquisition and the signal measurement, is surely negligible with respect to the preamp one.

The measurement is carried out by acquiring a single sample at the output of the preamplifier. Considering a preamp with a unitary DC gain and assuming to be able to collect the signal at its peak, i.e. for $T_P < t_m < 2T_P$, the collected signal is

$$S = V_P.$$

The noise can be computed by recalling that the equivalent bandwidth for noise computation at the output of a low-pass filter featuring a single pole is $\sigma = \sqrt{S_{B,OUT} * 2f_n}$, where $S_{B,OUT}$ is the bilateral spectral density at the output of the preamp and f_n is the equivalent noise bandwidth corresponding to the autocorrelation in zero of the weighting function divided by 2. In this case the weighting function h(t) is the single-pole low pass filter provided by the preamplifier. We can consider the preamplifier having unitary DC gain as this choice has no impact on the SNR. In this case, $S_{B,OUT} = S_{B,IN}$ (because of the unitary DC gain), $S_{B,IN} = S_V/2$ and recalling that $\mathrm{f_n} = \frac{\mathrm{k_{hh}(0)}}{2} = \frac{\pi}{2}\mathrm{f_{PA}}$, the noise expression can be rewritten as

$$N = \sqrt{S_V * \frac{\pi}{2}\mathrm{f_{PA}}}$$

leading to

$$\frac{S}{N} = \frac{V_P}{\sqrt{S_V * \frac{\pi}{2}\mathrm{f_{PA}}}}.$$

By definition, the minimum amplitude of a signal is the one that is necessary to have SNR=1. Therefore, the minimum amplitude that can be measured in this case without any additional filter is

$$V_{P,MIN} = \sqrt{S_V * \frac{\pi}{2}\mathrm{f_{PA}}} = 177\mu V.$$

The second request of this problem concerns the design of the ideal filter that maximizes the SNR and the computation of the corresponding minimum signal amplitude that can be recovered. Since the signal is only mixed with stationary white noise, the best possible filter is represented by the matched filter, i.e. a filter featuring a weighting function with the same shape of the signal itself. It could be noted that the matched filter could be either applied before or after the preamplifier, since the preamp does not affect the signal shape and the output noise is still wideband, i.e. its autocorrelation width is negligible with respect to the duration of the weighting function to be applied.
Considering a weighting function

$$b(t) = \begin{cases} \frac{1}{4} & for\ 0 < t < T_P \\ 1 & for\ T_P < t < 2T_P \\ \frac{1}{4} & for\ 2T_P < t < 3T_P \end{cases}$$

we have $s(t) = V_P * b(t)$. In this case, the optimum SNR is given by

$$\left(\frac{S}{N}\right)_{OPT} = \frac{V_P}{S_B}\sqrt{k_{bb}(0)} = \frac{V_P}{S_B}\sqrt{\int_{-\infty}^{+\infty} b^2(\alpha)d\alpha}.$$

As previously discussed, $S_B = S_V/2$ while $k_{bb}(0)$ can be computed by the sum of three contributions:

$$k_{bb}(0) = \int_0^{T_P}\left(\frac{1}{4}\right)^2 d\alpha + \int_{T_P}^{2T_P} d\alpha + \int_{2T_P}^{3T_P}\left(\frac{1}{4}\right)^2 d\alpha = \frac{9}{8}T_P.$$

As result,

$$\left(\frac{S}{N}\right)_{OPT} = \frac{V_P}{S_B}\sqrt{\frac{9}{8}T_P}.$$

By applying the definition, the minimum signal amplitude that could be ideally recovered with the optimum filter is

$$V_{P,MIN} = \sqrt{\frac{S_V}{2}\frac{8}{9T_P}} = 4.2\mu V.$$

Comment: the minimum amplitude that can be measured without filtering is almost two orders of magnitude higher than the ideal result that could be achieved with the given signal shape and noise contribution. For this reason, there is plenty of room for improvement in the given acquisition chain, consisting only of the preamp so far.

B) *Consider now to employ filters with variable parameters. Select a suitable practical filter, select its parameters to maximize the Signal-to-Noise ratio (S/N) and evaluate the minimum amplitude $V_{P,MIN}$ that can be measured in these conditio*ns.

Variable parameters filter feature a weighting function that varies over time; they can be exploited to acquire the input only when the signal is present, thus also acquiring noise only when strictly necessary, with beneficial effects on the SNR. It is worth noting that acquiring the entire signal is not necessarily convenient, but it depends on the peculiar shape of the signal.
The actual exploitation of variable-parameter filters requires precise knowledge of the absolute time of arrival of the signal or an auxiliary synchronization signal to enable the acquisition only when the signal is present. In this problem, the availability of a sync signal is not declared, but the employment of a variable-parameter filter is specifically demanded. To understand the meaning of this request, it is worth saying that in a real case it is common to evaluate the potential advantages of a solution, especially if it's a simple one, before facing the challenges of its actual implementation.

A suitable practical filter for the signal of interest is the gated integrator (GI) which can be used to implement a rectangular weighting function. To maximize the SNR, both the position and the duration of the single integration window must be properly chosen. Concerning the position, it is clearly convenient to have an acquisition window that includes the peak of the signal. On the contrary, it is not straightforward to understand what is the best duration of the integration window, which requires to understand whether it is convenient or not to acquire the signal where its amplitude is one fourth of the peak amplitude. Such problem must be mathematically solved by maximizing the SNR as a function of the duration of the integration window. Exploiting the symmetrical shape of the signal, we can use a GI centered around the signal, i.e. at $1.5T_P$ from the beginning of the signal itself, and we can express its duration as $T_G = T_P + 2x$, with $0 < x < T_P$. For the sake of simplicity, we'll consider a unitary amplitude of the weighting function. The weighting function is sketched in Fig. 1.1.

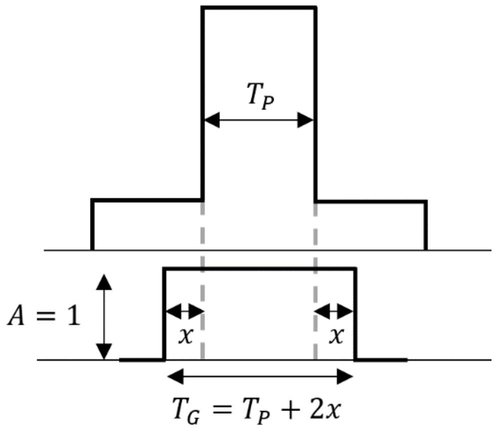


Figure 1.1: Weighting function of a GI with respect to the signal.

In this scenario, both signal and noise can be expressed as a function of x.
The signal is given by the following expression:

$$S(x) = V_P T_P + \frac{V_P}{4} 2x = V_P * (T_P + \frac{x}{2})$$

While the noise is

$$N(x) = \sqrt{S_{B,OUT} * A^2 T_G} = \sqrt{\frac{S_V}{2} * (T_P + 2x)}$$

Leading to a signal to noise ratio featuring the following form:

$$SNR(x) = \frac{V_P}{\sqrt{\frac{S_V}{2}}} \frac{T_P + \frac{x}{2}}{\sqrt{(T_P + 2x)}}$$

In order to maximize $SNR(x)$ we can compute its derivative as follows:

$$\frac{\delta SNR}{\delta x} = \frac{V_P}{\sqrt{\frac{S_V}{2}}} \frac{\frac{1}{2}\left(\sqrt{T_P + 2x}\right) - \frac{(T_P + \frac{x}{2})}{\sqrt{T_P + 2x}}}{T_P + 2x} = \frac{V_P}{\sqrt{\frac{S_V}{2}}} \frac{-\frac{1}{2}T_P + \frac{x}{2}}{(T_P + 2x)^{1.5}}$$

The derivative of this function goes to zero for $x = T_P$. Nevertheless, such value of x does not correspond to a maximum of the SNR. This can be easily verified by computing the SNR coefficient $\alpha(x) = \frac{T_P + \frac{x}{2}}{\sqrt{(T_P + 2x)}}$ for $x = 0$ and for $x = T_P$.

$$For\ x = 0, \qquad \alpha(x) = \sqrt{T_P}$$

$$For\ x = T_P,\ \alpha(x) = \frac{\frac{3}{2}T_P}{\sqrt{3T_P}} = \frac{\sqrt{3}}{2}\sqrt{T_P} = 0.866\sqrt{T_P}.$$

Since $SNR(x = T_P) < SNR(x = 0)$, $x = T_P$ can't correspond to a maximum of the function, so it must lead to a minimum.

Extra: for the sake of completeness, the plot of $SNR(x)$ is reported in Fig. 1.2.

$$SNR(x) = \frac{10 + \frac{x}{2}}{\sqrt{10 + 2x}}$$

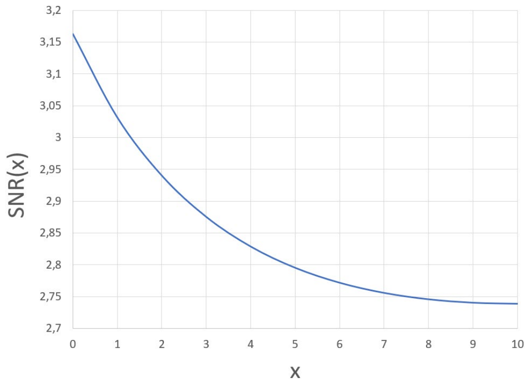


Figure 1.2: SNR as a function of the variable x. The y-axis unit is arbitrary.

In summary, the SNR is maximized for $x = 0$, which corresponds to a GI with $T_G = T_P$ and centered around the signal peak. The parameters of the GI are:

- Start of the integration window: $t = T_P$.
- Duration of the integration window: $T_G = T_P$.
- Weight of the filter: $A = 1$.

With this filter and the listed parameters, we obtain

$$\frac{S}{N} = \frac{V_P}{\sqrt{\frac{S_V}{2}}}\sqrt{T_P}.$$

and by applying the definition, we get

$$V_{P,MIN} = \frac{\sqrt{S_V}}{\sqrt{2T_P}} = 4.47\mu V.$$

Comment: it is worth noting that the minimum amplitude that can be recovered with a Gated Integrator is only slightly higher (6.4%) than the one that could be theoretically achieved with the matched filter, meaning that the GI with the selected parameters is a very good practical solution in this case.

C) *Consider now to follow a fully digital approach. Discuss the guidelines to select the sampling frequency and how this choice could have an impact on the Signal-to-Noise ratio. Choose a reasonable value for the sampling frequency and evaluate the minimum amplitude* $V_{P,MIN}$ *that can be measured in this case.*

With a digital approach, it is possible to implement a great variety of weighting functions. Above all, this approach can allow the implementation of a filter having the same (discrete) shape of the optimum one. In this case, we have a signal with a finite duration and a variable amplitude. By imitation of the optimum filter, we can apply a weight that is proportional to the amplitude of each sample. For example, we can apply a unitary weight to the signal peak samples (for any $T_P < t < 2T_P$) and a weight of $\frac{1}{4}$ to the rest of the samples acquired between 0 and T_P and between $2T_P$ and $3T_P$. Concerning the sampling frequency, there is a tradeoff between performance and complexity of the system. Indeed, the higher the frequency, the higher the number of samples that are collected and exploited, with beneficial effects on the SNR; at the same time, the higher the frequency, the higher is the complexity of the system. The study carried out in point A regarding the optimum filter sets the benchmark also for the digital filter design: if we used an infinite sampling frequency, we would obtain the same result of the optimum filter. On the other hand, to have an idea of what could be obtained with a finite sampling time, we can consider a sampling frequency that guarantees the uncorrelation of noise samples. In this way, it is possible to get a simple pencil-and-paper estimation of the result that could be achieved with a feasible digital approach,. To do so, we need to evaluate the minimum time distance between uncorrelated noise samples. Considering the initial acquisition scheme, composed only by the signal source and the preamplifier, the white noise is only limited by the pole of the preamplifiers, leading to a Lorentzian spectrum $S_{OUT}(f) = \frac{\frac{S_V}{2}}{1+(\frac{f}{f_{PA}})^2}$, corresponding to an exponential autocorrelation function $R_{nn}(\tau) = \sigma^2 e^{-|\tau| 2\pi f_{PA}}$. As a result, two noise samples are substantially uncorrelated if their mutual time distance is at least equal to $\frac{5}{2\pi f_{PA}} = 15.9ns$.

By choosing a sampling time $T_{SAMPLE} \geq 15.9ns$ we can easily calculate the SNR because noise samples are practically uncorrelated. For example, we can choose $T_{SAMPLE} = 16ns$. In this scenario the signal is given by

$$S = V_P \frac{T_P}{T_{SAMPLE}} + 2 * \frac{V_P}{4} \frac{1}{4} \frac{T_P}{T_{SAMPLE}}$$

While the noise can be computed with the following expression:

$$N = \sqrt{S_V * \frac{\pi}{2} f_{PA} (\left(\frac{1}{4}\right)^2 \frac{2T_P}{T_{SAMPLE}} + 1^2 * \frac{T_P}{T_{SAMPLE}})}$$

It could be noted that the SNR can be seen as the SNR of a single sample on the peak, i.e.

$$SNR_{1sample} = \frac{V_P}{\sqrt{S_V * \frac{\pi}{2} f_{PA}}},$$

enhanced by a factor that is equal to the square root of an equivalent number of samples

$$\sqrt{N_{EQ}} = \sqrt{\frac{1}{16} \frac{2T_P}{T_{SAMPLE}} + \frac{T_P}{T_{SAMPLE}}}.$$

As a result, the minimum amplitude that can be recovered with a single sample, $V_{P,MIN} = \sqrt{S_V * \frac{\pi}{2} \mathrm{f_{PA}}} = 177\mu V$ as computed in point A, is reduced by $\sqrt{N_{EQ}} = \sqrt{703}$. In conclusion,

$$V_{P,MIN} = \frac{\sqrt{S_V * \frac{\pi}{2} f_{PA}}}{\sqrt{N_{EQ}}} = \frac{177\mu V}{\sqrt{703}} = 6.67\mu V.$$

Comment: by increasing the sampling frequency, it is possible to approach the analog theoretical result obtained with the matched filter.

D) *Consider now an additional 1/f noise component with a corner frequency* $f_C = 5kHz$ *affecting the preamplifier. Discuss the impact of this additional noise component on the final S/N and propose at least two different solutions to minimize this effect. Choose one of the two proposed solutions, provide quantitative data (e.g. filter parameters) and evaluate the minimum amplitude* $V_{P,MIN}$ *that can be measured in these conditions.*

The additional 1/f noise component would theoretically cause the divergence of noise, if no limitation at low frequency is present. As briefly discussed in the solution of point A), in a real case such limitation always exists, at least given by the zero setting of any instrument that could be used to acquire the signal. Nonetheless, we are asked to propose at least two different solutions to minimize the impact of 1/f noise on this measurement. Clearly, both solutions must involve a high-pass filtering action.
After discussing the two solutions, the problem asks us to provide quantitative computation just for one of them. However, a quantitative analysis with **two** appropriate options for this problem is provided here for the sake of completeness.

A simple high-pass filter consists of a CR network. In order to preserve the shape of the signal, thus keeping as much signal as possible and preserving the possibility of using the GI sized in point B), we need to size the time constant of the CR filter to be much longer than T_P. For example, we can choose $\tau_{CR} = 100T_P = 1ms$, corresponding to a high-pass filtering action with $f_{P,CR} = 159Hz$. By keeping the same GI of point B, we can approximate the high-frequency cut-off with $f_{LP} \approx \frac{1}{2T_G} = 50kHz$. It is worth recalling that a coarse estimation of bandwidth limitations is adequate for 1/f noise computation as this depends on the logarithm of the bandwidth ratio $\frac{f_{LP}}{f_{HP}}$. By considering a GI with unitary DC gain, we can compute the 1/f contribution

$$\sigma_{1/f} = \sqrt{S_V f_C \, ln\left(\frac{f_{LP}}{f_{P,CR}}\right)} = 3.39\mu V.$$

Such contribution must be quadratically summed to the white noise contribution computed in point B) (after rescaling the white noise contribution by a factor $(\frac{1}{T_G})^2$ for consistency with GI applied to 1/f noise), thus obtaining

$$V_{P,MIN} = 5.6\mu V.$$

Comment: with a mild but simple high-pass filtering action, selected to avoid any degradation of the signal shape, the minimum amplitude that can be recovered has been increased by approximately 25% with respect to the case without 1/f noise. This result represents a good starting point and it could pave the way to an optimization of this solution, depending on the real application requirements and constraints.

An alternative approach to apply a high-pass filtering action on noise is represented by the Correlated Double Filtering (CDF), having the great advantage of avoiding any impact on the signal. In this case, the GI sized in point B) could be re-used, combined with an analog or a digital differentiator. Since no repetition rate is reported among data, we can assume that the signal is not repeated over time. Therefore, there is plenty of time to integrate the noise in absence of signal, thus avoiding noise doubling. At the same time, the longer the duration of the integration window, the lower is its high-pass filtering action. The simplest solution for a fast computation consists in using a long integration window, like $T_{G,noise} = 10T_P$. The window for noise sampling must be placed before the beginning of the signal, i.e. at a center-to-center time distance of $\Delta T = 5T_P + T_P + 0.5T_P = 6.5T_P$ from the principal GI, as sketched in Fig. 1.3. Considering a unitary DC gain for both integration windows, the signal to noise ratio can be estimated as follows:

$$\frac{S}{N} \approx \frac{V_P}{\sqrt{S_V \frac{1}{2T_G} + S_V f_c ln\,(\frac{2\pi\Delta T}{2T_G})}} = \frac{V_P}{\sqrt{S_V}\sqrt{\frac{1}{2T_G} + f_c ln\,(\frac{2\pi\Delta T}{2T_G})}}$$

$$= \frac{V_P}{\sqrt{S_V}\sqrt{50kHz + 5kHz * 3.2}}$$

Consequently,

$$V_{P,MIN} = 5.14\mu V.$$

Comment: the dominant contribution to noise with the CDF is given by the white noise. Therefore, using a large integration window to avoid the doubling of that noise was indeed a good solution.

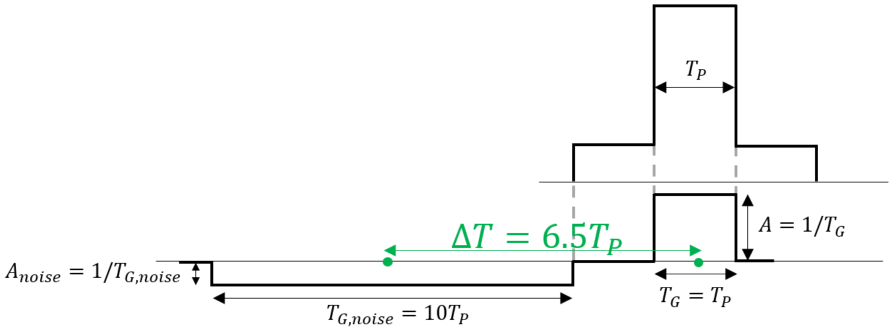


Figure 1.3: Correlated double filtering (CDF) applied to the signal of interest.

PROBLEM 2

June 17th, 2019 – Pb. 1

In this exercise, the information of interest is encoded in the amplitude of multiple triangular pulses. The exploitation of one single pulse in presence of only white noise is considered first. Then, multiple pulses either having fixed or random inter-arrival time is discussed in the presence of also 1/f noise.

A) Evaluate the minimum measurable amplitude $V_{P,MIN}$ of the signal shown in Fig. 1 without using any kind of filter. Consider now to employ a gated integrator, select its parameters for maximizing the Signal-to-Noise ratio (S/N) and evaluate the minimum measurable amplitude$V_{P,MIN}$.

Now consider the case in which a series of pulses with a random arrival time arrive at the preamplifier. The mean value of the arrival time is 50 µs. We want to measure the amplitude of each individual pulse. The measurement takes about 8 hours. Also 1/f noise component is present in the amplifier with a frequency corner of 10 kHz.

B) Discuss how much 1/f component has an impact on the final S/N and how to minimize this effect. Calculate the new final S/N.

C) Now consider the case in which the pulses arrive periodically with a period equal to 50µs. How does the answer change to the previous point?

D) The amplitude of the individual pulses changes slowly with a time scale around 1s. Assuming you are no longer interested in measuring the single pulse, how can you exploit this new information? How does the signal to noise ratio improve? Provide a quantitative evaluation.

Pulse signal
V_P variable pulse amplitude
A sync signal is provided for each pulse
Preamplifier
$\sqrt{S_V} = 10nV/\sqrt{Hz}$ white noise spectral density (unilateral)
$f_{PA} = 100MHz$ upper band-limit

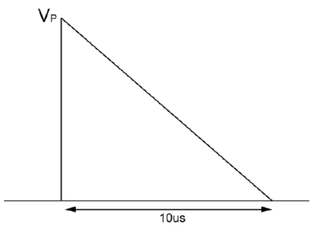


Fig.1

SOLUTION

In this problem, the waveform of the signal of interest is graphically fully described in Fig. 1. To begin with, we can write the mathematical expression of the signal:

$$s(t) = V_P\left(1 - \frac{t}{T_P}\right) for\ 0 \le t \le T_P$$

where t=0 is chosen to coincide with the starting point of the signal.
The duration of the signal is $T_P = 10\mu s$.
The signal is extracted by means of a preamplifier featuring a single pole at $f_{PA} = 100MHz$**.** A rough estimation of the signal bandwidth allows us to rapidly evaluate the effect of the preamplifier on the signal shape. To this aim, we can overestimate the signal frequency content by simply approximating it with a rectangular pulse of duration T_P. The Fourier transform of the rectangular signal is a cardinal sine function having the first zero-point in $f = 1/T_P$**.** Since here $\frac{1}{T_P} = 100kHz$, that is three order of magnitudes lower than the preamplifier pole, we can assess that the signal shape is substantially unchanged at the output of the preamplifier with no need to refine our estimation.

Alternative procedure*:* the signal bandwidth could be also roughly estimated by approximating the triangular shape with an exponential signal $e(t) = V_P e^{-\frac{t}{\tau}}$ having its maximum slope equal to the constant slope of the signal of interest. The maximum slope of the exponential signal is V_P/τ in t=0 while the triangular signal has a fixed slope of V_P/T_P. Equating these two slopes, we can roughly approximate the real signal with an exponential signal having $\tau = T_P$. The Fourier transform of this equivalent exponential signal has a Lorentzian shape limited by a pole at $\frac{1}{2\pi\tau} = 15{,}9kHz$. Also with this approximation we verified that the frequency content of the signal is concentrated in a range of frequencies that is transmitted by the preamplifier without any attenuation.
The significant difference – about a factor 6 - between the two estimated values is due to the fact that the rectangular approximation is quite conservative, while the exponential approximation provides a better fit of the actual signal shape.

A) *Evaluate the minimum measurable amplitude $V_{P,MIN}$ without using any additional filter. Then, describe and explain the ideal filter that makes it possible to measure the pulse amplitude V_P with the best possible Signal-to-Noise ratio and evaluate the minimum amplitude $V_{P,MIN}$ thus measurable.*
The first question that we need to address in this problem concerns the sensitivity that can be achieved without any additional filter following the preamplifier. As widely discussed above, the shape of the signal at the preamplifier output is the

same triangular shape that is reported in Fig. 1. By exploiting the available sync signal we can maximize the signal collection with a single-shot measurement. By sampling the signal at t=0 where the signal amplitude is maximum and considering a preamp with unitary gain, we obtain:

$$S = V_P$$

At the output of the preamplifier, the signal is superimposed to wideband noise only filtered by the preamplifier itself that is a constant-parameter low-pass filter.

The transfer function of the preamplifier is $H(f) = \frac{1}{1+j2\pi f T_{pa}}$ and consequently

$$|H(f)|^2 = \left|\frac{1}{1 + j2\pi f \frac{1}{2\pi f pa}}\right|^2 = \frac{1}{1+\left(\frac{f}{f pa}\right)^2}.$$

Therefore, the power spectral density at the output of the preamplifier is $S_{out}(f) = S_B \cdot |H(f)|^2 = \frac{S_V}{2} \frac{1}{1+\left(\frac{f}{f pa}\right)^2}$.

By integrating the output noise spectrum, we obtain:

$$N = \int_{-\infty}^{+\infty} S_{out}(f)df = \int_{-\infty}^{+\infty} \frac{S_V}{2} \frac{1}{1+\left(\frac{f}{f pa}\right)^2} df = \frac{S_V}{2} f_{pa} \left[arctan\left(\frac{f}{f_{pa}}\right)\right]_{-\infty}^{+\infty} =$$

$$\frac{S_V}{2} f_{pa} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = S_V \frac{\pi}{2} f_{pa} = 125\mu V.$$

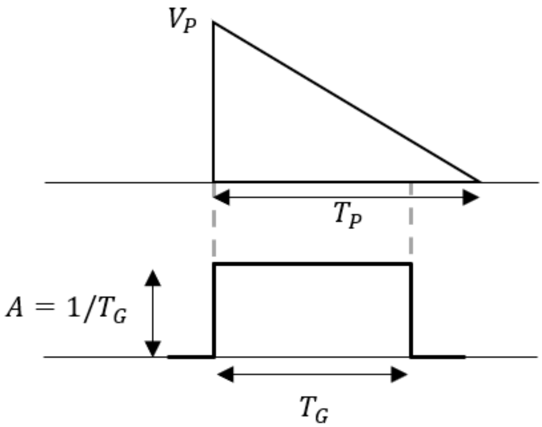
The minimum measurable amplitude of V_P is by definition the value that produces a SNR=1. Therefore

$$\frac{S}{N} = \frac{V_P}{\sqrt{S_V \frac{\pi}{2} f_{pa}}} = 1$$

$$V_{P,MIN} = 125\mu\text{V}$$

The second request of this problem concerns the exploitation of a gated integrator with optimized parameters to maximize the obtainable SNR.
Two parameters of the gated integrator affect the SNR, i.e. its duration (T_G) and its position with respect to the signal, while the filter weight A plays no role in the SNR. A qualitative evaluation of the signal is sufficient to choose the starting time of the GI integration window. Indeed, it is convenient to start the integration at the beginning of the signal where it features its maximum amplitude. The integration will last for at most the entire duration of the signal, since an acquisition where the signal is not present would only increase the noise with clear detrimental effects on the SNR. Therefore, $0 < T_G < T_P$.

Considering $A = \frac{1}{T_G}$ and the aforementioned alignment between the signal and the integration window, we obtain:



The weighting function of the GI: $w_{GI} = \frac{1}{T_G} rect[0, T_G]$

that is to say $w_{GI} = \begin{cases} \frac{1}{T_G}\ for\ 0 < t < T_G \\ 0\ somewhere\ else \end{cases}$

As a result, the signal collected by the GI is:

$$S_{GI} = \int_{-\infty}^{+\infty} S_{IN}(t) w_{GI}(t) dt = \int_{-\infty}^{+\infty} V_P \left(1 - \frac{t}{T_P}\right) \times \frac{1}{T_G} rect[0, T_G] dt = \frac{V_P}{T_G} \int_0^{T_G} \left(1 - \frac{t}{T_P}\right) dt = \frac{V_P}{T_G} \left[t - \frac{t^2}{2T_P}\right]_0^{T_G} = V_P \left(1 - \frac{T_G}{2T_P}\right)$$

The noise integrated by the filter can be computed considering that the noise spectrum at the output of the preamplifier is limited by a pole that is much higher with respect to most of the frequency content of the GI (the first zero of the GI in frequency is at $\frac{1}{T_G}$ that ranges from a few MHz for $T_G < 1\mu s$ down to 100kHz for $T_G = T_P$). Consequently, the noise at the input of the GI can be considered white, thus simplifying the noise computation as follows:

$N = \int_{-\infty}^{+\infty} S_B k_{ww,GI}(t) dt \cong S_B k_{ww,GI}(0) = \frac{S_V}{2} k_{ww,GI}(0)$

The autocorrelation of the weighting function is:

$k_{ww,GI} = \int_{-\infty}^{+\infty} w_{GI}^2(t) dt = \int_{-\infty}^{\infty} \frac{1}{T_G^2} rect^2[0, T_G] dt = \int_0^{T_G} \frac{1}{T_G^2} dt = \frac{1}{T_G^2} [t]_0^{T_G} = \frac{1}{T_G}$

As a result, $N_{GI} = S_V \frac{1}{2T_G}$.

The expression of the SNR as a function of T_G is:

$$\left(\frac{S}{N}\right)_{GI} = \frac{V_P\left(1-\frac{T_G}{2T_P}\right)}{\sqrt{S_V\frac{1}{2T_G}}}$$

which can be rewritten as

$$\left(\frac{S}{N}\right)_{GI} = \frac{V_P}{\sqrt{\frac{S_V}{2T_P}}}\frac{\sqrt{T_G}}{\sqrt{T_P}}\left(1-\frac{T_G}{2T_P}\right).$$

In order to maximize the SNR, we will now introduce the variable $x = \frac{T_G}{T_P}$.

The SNR is a function $f(x)$ having the following expression.

$\left(\frac{S}{N}\right)_{GI} = \frac{V_P}{\sqrt{\frac{S_V}{2T_P}}}\sqrt{x}\left(1-\frac{x}{2}\right) = f_0 \cdot \sqrt{x}\left(1-\frac{x}{2}\right)$, where f_0 summarizes the terms that do not depend on x.

To maximize the SNR we need to find the maximum of $g(x) = \left(1-\frac{x}{2}\right)\sqrt{x}$.

By deriving the function we obtain $\frac{dg(x)}{dx} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{4}$ that is equal to 0 for $x = \frac{2}{3}$, that is for $T_G = \frac{2}{3}T_P$.

It should be verified that the computed value of x represents a maximum and not a minimum (in both cases the first derivative would be zero).

It can be observed that, for $T_G = 0$, the SNR is equal to 0, while for $T_G = \frac{2}{3}T_P$ the SNR>0. Considering that the computed value of x is the only one that nullifies $g'(x)$ in the admissible range of T_G values and $g(\frac{2}{3}) > g(0)$, $x = \frac{2}{3}$ cannot be a minimum of the function and it is therefore a maximum.

An alternative way to get to same conclusion consists in computing $g(x)$ for a small set of values around $x = \frac{2}{3}$, thus verifying that $f\left(x = \frac{2}{3}\right)$ provides the highest result.

It is worth recalling that the general mathematical approach to discriminate between maximum and minimum values of a function consists in computing the second derivative of the function. Since

$$\frac{d^2g(x)}{dx^2} = -\frac{1}{4}\frac{1}{\sqrt{x^3}} - \frac{3}{4}\frac{1}{2\sqrt{x}}$$

is lower than zero for any admissible value of x, the concavity of the function is downward and thus the value that nullifies the first derivative is a maximum. For the sake of completeness, the waveform of $g(x)$ on an arbitrary scale is here reported.

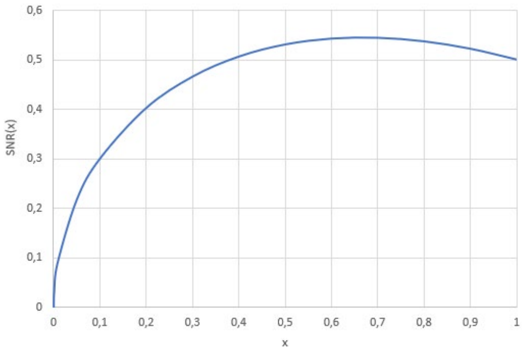


Figure 2.1: *SNR as a function of the variable x. The y-axis unit is arbitrary*

After optimizing the duration of the integration window, we need to compute the resulting SNR with a gated integrator featuring $T_G = \frac{2}{3}T_P$, $A = \frac{1}{T_G}$ and starting the integration at the beginning of the signal.

We already computed the SNR before, so we only need to substitute the computed value of T_G obtaining:

$$\left(\frac{S}{N}\right)_{GI} = \frac{V_P}{\sqrt{\frac{S_V}{2T_P}}}\frac{\sqrt{T_G}}{\sqrt{T_P}}\left(1-\frac{T_G}{2T_P}\right) = \frac{V_P}{\sqrt{\frac{S_V}{2T_P}}}\sqrt{\frac{2}{3}}\left(1-\frac{2}{6}\right) = \frac{V_P}{\sqrt{\frac{S_V}{2T_P}}}\sqrt{\frac{2}{3}}\left(\frac{2}{3}\right)$$

In this case, the minimum measurable amplitude of V_P is:

$$V_{P,MIN,GI} = \frac{\sqrt{\frac{S_V}{2T_P}}}{\frac{2}{3}\sqrt{\frac{2}{3}}} = 4{,}1\ \mu V.$$

Comment: the availability of the sync signal has allowed the exploitation of a gated integrator. Tailored on the specific features of the signal of interest, that has a known shape, the low-pass filtering action of the GI improved the SNR by more than a factor of 40.

New scenario:

Now consider the case in which a series of pulses with a random arrival time arrive at the preamplifier. The mean value of the arrival time is 50 µs. We want to measure the amplitude of each individual pulse. The measurement takes about 8 hours. Also 1/f noise component is present in the amplifier with a frequency corner of 10 kHz.

From now on, a new scenario is to be considered. Three key additional information are presented at this point:

1) Concerning the signal, the text states that there are multiple pulses. Nevertheless, the information of interest relies in the amplitude of each single pulse, that is to be measured individually. In this scenario, the presence of multiple pulses does not increase the exploitable information to increase the SNR with respect to the single pulse, but, on the contrary, it must be carefully handled in order to avoid mixing up different information. The random arrival time of pulses, where only the mean interarrival time (50μs) is known, poses a challenge in keeping the information of different pulses separate from one another.
2) The duration of the measurement is of about 8 hours. This means that the system cannot be paused and/or reset during this time framework. From a practical point of view, a zero setting of the instruments can only occur before a 8-hour measurement is started.
3) Concerning the noise, from now on we have to consider an additional 1/f noise component, with corner frequency of 10kHz. Ideally, 1/f noise has a divergent spectrum towards low frequencies.

B) *Discuss how much 1/f component has an impact on the final S/N and how to minimize this effect. Calculate the new final S/N.*

The presence of 1/f noise requires some limitation to low frequencies, otherwise the noise contribution would be extremely high (infinite if we consider an ideal case). In a real case, an high pass filtering action is always provided by a measurement system by means of a zero-setting of the instrument. To have an idea of the high-pass filtering action that a zero-setting can provide we will consider the following scenario:

- the zero setting is applied right before starting the 8-hour measurement;
- the integration window of the zero setting is quite long, i.e. much longer than any weighting function that is applied to the signal.

The first assumption implies that the worst case scenario for the high-pass filtering action is the one having the pulse of interest at a distance of about 8 hours from the zero setting integration window. On the other hand, having a long integration window prevents noise doubling.
Since it is not specified, we can consider the possibility of exploiting the same gated integrator that we sized in point 1. In this case, the low-pass filtering action on 1/f noise can be estimated with the same equivalent bandwidth of white noise, that is $f_{LPF} \cong f_n = 1/2T_G$. The high-pass filtering action can be estimated considering a time distance T_W of about 8 hours between the zero setting and the gated integrator, i.e. $f_{HPF} \cong 1/2\pi T_W$. The 1/f noise contribution is computed as follows:

$$\sigma_{1/f} = \sqrt{S_V f_c \, ln\left(\frac{f_{LPF}}{f_{HPF}}\right)} = \sqrt{S_V f_c \, ln\left(\frac{\frac{1}{2T_G}}{\frac{1}{2\pi T_W}}\right)} = 4{,}8 \; \mu V$$

White noise contribution is the same of point A, where a GI with amplitude $1/T_G$ and duration $T_G = 2/3 \; T_P$ was used:

$$\sigma_W = \sqrt{S_V \frac{1}{2T_G}} = \sqrt{S_V \frac{3}{4T_P}} = 2{,}7 \; \mu V$$

The overall noise contribution is mainly due to the 1/f noise:

$$N = \sqrt{\sigma_{1/f}^2 + \sigma_W^2} = 5{,}5 \; \mu V$$

As a result, the sensitivity of the system is worsened with respect to point A.

$$V_{P,MIN} = \frac{5{,}5 \; \mu V}{\frac{2}{3}\sqrt{\frac{2}{3}}} = 10{,}1 \; \mu V.$$

To reduce the dominant contribution, the overall high pass filtering action of the system should be improved. Unfortunately, the randomness of the interarrival time of pulses prevents the exploitation of a simple constant-parameter filter such as the CR high-pass filter. Indeed, the output of the constant-parameter differentiator would be affected by fluctuations due to variations in the interarrival time of pulses.

To overcome this issue, a switched parameter filter can be used. Starting from the simple CR network, we can add a switch to implement a baseline restorer (BLR). Thanks to the exploitation of the sync signal, it is possible to integrate the noise only when the signal is not present. This possibility has a twofold advantage: on one hand, the high pass filtering action is applied only to noise, thus avoiding any signal loss due to the additional filtering stage; on the other hand, the noise acquisition is completely independent from the pulse repetition rate. Indeed, every time a pulse is present, the noise acquisition is paused and then it is restarted without any change at the end of the pulse acquisition.

The BLR can be added to the acquisition chain following the preamplifier and preceding the gated integrator. In this case, to avoid noise doubling, the overall duration of the BLR weighting function ($T_{BLR} = 5\tau_{BLR}$) can be chosen to last for at least 10 times the GI integration window, i.e. $T_{BLR} \geq 10T_G$.

Moreover, to simplify the acquisition scheme, a GI integration window equal to the pulse duration ($T_G = T_P$) could be considered. In this scenario, the same control signal derived from the sync could be used to control both the BLR, opening the switch when the signal is present, and the GI, closing the switch to integrate the entire pulse.

To roughly estimate the equivalent high-pass filtering action of the BLR, we need to have an idea of the overall duration of the BLR weighting function including when it is paused by pulse arrival. With $T_{BLR} = 10T_G = 10T_P = 100\mu s$, on average we would have a couple of pulses within T_{BLR}, thus increasing the effective duration of this function by about $2T_P = 20\mu s$. As a result, $T_{BLR,effective} \approx 120\mu s$.

As for the correlated double filtering, we can estimate the equivalent high pass filtering action by using time distance T_D between the center of the GI applied to the pulse and the center of the baseline acquisition window. Since the baseline is acquired by means of an exponential weighting function in the BLR, we can approximate its center with τ_{BLR}. In our case, $\tau_{BLR} = 20\ \mu s$. To be conservative, we can increase this value by T_P assuming that one pulse will sometimes occur within τ_{BLR}. As a result, $T_D = \frac{T_G}{2} + \tau_{BLR} + T_P$. Choosing $T_G = T_P$ to minimize the implementation complexity as discussed above, we would obtain $T_D = \frac{3T_G}{2} + \tau_{BLR} = 35\mu s$. Considering a gated integrator with A=$1/T_P$, we get:

$$S = \frac{V_P T_P}{2}\frac{1}{T_P} = \frac{V_P}{2}$$

$$\sigma_W \cong \sqrt{S_V\left(\frac{1}{2T_P} - \frac{1}{2\pi T_D}\right)} = 2{,}13\ \mu V$$

$$\sigma_{1/f} \approx \sqrt{S_V f_c\, ln\left(\frac{\frac{1}{2T_P}}{\frac{1}{2\pi T_D}}\right)} = \sqrt{S_V f_c\, ln\left(\frac{\frac{1}{20\mu s}}{\frac{1}{2\pi(35\mu s)}}\right)} = 1{,}55\ \mu V$$

The overall noise contribution is now mainly due to the white noise contribution:

$$N \approx \sqrt{\sigma_{1/f}^2 + \sigma_W^2} = 2{,}63\ \mu V$$

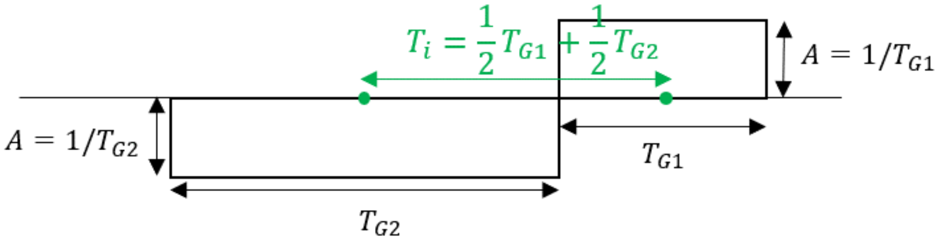
Resulting into a reduced minimum measurable value of V_P:

$$V_{P,MIN} = 2 * 2{,}63\ \mu V = 5{,}26\ \mu V$$

Comment: the exploitation of the baseline restorer proved effective to significantly improve the high-pass filtering action on the noise while overcoming the issue of the random inter-arrival time of pulses.

C) *Now consider the case in which the pulses arrive periodically with a period equal to 50µs. How does the answer change to the previous point?*

In this point there is a substantial difference with respect to point B), that is that the interarrival time between pulse pairs (from now on referred to as T_R) is fixed and known. This opens the way to the exploitation of two different solutions with respect to the previous case:
1)a CR differentiator as there would be no fluctuations of the baseline in this case;
2)a correlated double filtering (CDF) since there is always enough time to acquire the baseline before each pulse.
The simplicity of the CR differentiator is counterbalanced by its action affecting not only the noise but also the signal. With a repetition rate that in this problem is equal to only five times the duration of the pulse, the CR would downshift the pulse amplitude by a factor that is around 1/6 (it would be exactly 1/6 if the pulse was a rectangle). Moreover, in this specific problem a correlated double filtering could be added with a very low level of complexity since a gated integrator has already been designed and proven effective in point A). In this scenario, the CDF is at the same time the simplest solution and the most advantageous one as it perfectly exploits the features of the signal of interest.
We will then proceed by measuring the amplitude of the triangular input signal by means of two acquisitions based on a gated integrator (GI), as illustrated below. The weighting function consists of:
(i) the GI that has already been discussed and sized in point A), having a duration of $T_{G,1} = \frac{2}{3} T_P$ and starting from the rising edge of the pulse.
(ii) another integration window for baseline acquisition which precedes the pulse and lasts for the entire time interval between pulse pairs, i.e. $T_{G,2} = T_R - T_P$.
By doing so, the centre-to-centre distance of the two integration windows – which is useful to estimate the high pass filtering action provided by the subtraction of the two integrated values – is equal to $T_i = \frac{1}{2}(T_{G,1} + T_{G,2})$.
It is worth noting that both integrating windows have the same area, thus allowing a direct subtraction of the two values to compute the output. Moreover, only one integration window $(T_{G,1})$ acts on the signal: since it is only positive, any subtraction applied to it would cause signal loss. Finally, the area of each integration windows is equal to 1 in order to simplify noise computation as will be highlighted later in this paragraph.



As previously remarked, the signal is only collected during $T_{G,1}$, that is sized as in point A). Therefore, we have the same signal that we had in point A), that is:

$$S = V_P\left(1 - \frac{T_{G,1}}{2T_P}\right) = \frac{2}{3}V_P$$

In order to calculate the noise contribution, we can exploit the fact that both integration windows have unitary area.

Concerning the low pass filtering action, we can use the equivalent noise bandwidth of the two gated integrators:

$f_{n,eq,GI,1} = \frac{1}{2T_{G,1}} = 75kHz$

$f_{n,eq,GI,2} = \frac{1}{2T_{G,2}} = 12{,}5kHz$

while for the high-pass filtering action we can roughly estimate it as

$f_{i,eq} \approx \frac{1}{2\pi T_i} = 6.8kHz.$

From theory we recall that the effect of the designed CDF on white and 1/f noise can be computed as follows:

$$\sigma_W \cong \sqrt{S_V[(f_{n,eq,GI,1} - f_{i,eq}) + (f_{n,eq,GI,2} - f_{i,eq})]} \cong 2{,}7\mu V$$

$$\sigma_{1/f} \approx \sqrt{S_V f_c\left[ln\left(\frac{f_{n,eq,GI,1}}{f_{i,eq}}\right) + ln\left(\frac{f_{n,eq,GI,2}}{f_{i,eq}}\right)\right]} \cong 1{,}7\mu V$$

Consequently, the overall noise contribution is:

$$N = \sigma_{TOT,CDF} \cong 3{,}2\mu V$$

and the minimum amplitude $V_{P,MIN}$ that could be measured with this solution is:

$V_{P,MIN} = \sigma_{TOT,CDF} * \frac{3}{2} = 4{,}8\ \mu V.$

Extra: for a fair comparison with the BLR exploited in point B, we should use a non-optimized integration window on the signal also in this case.

By using $T_{G,1} = T_P = 10\mu s$, we obtain $S = \frac{V_P}{2}$, and the center-to-center distance between the two integration windows is slightly changed, i.e. $T_i = \frac{1}{2}(T_{G,1} + T_{G,2})= 25\mu s$. As a result, $f_{n,eq,GI,1} = \frac{1}{2T_{G,1}} = 50kHz$ and $f_{i,eq} \approx \frac{1}{2\pi T_i} = 6{,}3kHz$. Recomputing the two noise contributions, we obtain

$$\sigma_W \cong \sqrt{S_V\left[\left(f_{n,eq,GI,1} - f_{i,eq}\right) + \left(f_{n,eq,GI,2} - f_{i,eq}\right)\right]} \cong 2{,}2\mu V$$

$$\sigma_{1/f} \approx \sqrt{S_V f_c\left[ln\left(\frac{f_{n,eq,GI,1}}{f_{i,eq}}\right) + ln\left(\frac{f_{n,eq,GI,2}}{f_{i,eq}}\right)\right]} \cong 1{,}66\mu V$$

Leading to an overall noise contribution of $\sigma_{TOT,CDF,2} \cong 2{,}7\mu V$.
In this case, $V_{P,MIN} = 2 * \sigma_{TOT,CDF,2} = 5{,}4\ \mu V$, which is very close to the result obtained with the BLR (the exact difference between the two approaches can't be quantified since both results have been obtained with a certain degree of approximation). In similar conditions these two solutions can basically provide the same filtering action, but only the BLR can be easily exploited with a random arrival time of pulses.

D) *The amplitude of the individual pulses changes slowly with a time scale around 1s. Assuming you are no longer interested in measuring the single pulse, how can you exploit this new information? How does the signal to noise ratio improve? Provide a quantitative evaluation.*

In this point it is not specified whether the scenario of point B), i.e. with random interarrival time, or the one of point C), that is with a fixed and known time interval between pulses, is to be considered. However, coming right after point C), we can assume that in point D) the scenario is the same of the previous point. In this case, the CDF applied to the single pulse in point C) can be extended to the larger number of pulses by summing up the output of multiple CDFs, but their contribution must be properly weighted to take into account that the signal is slowly varying over time. Indeed, if we focus on a pulse at a certain time instant $\overline{t_m}$, similar information on its amplitude is present on all pulses that occur in a time window of 1s before$\overline{t_m}$; however, pulses occurring very close to $\overline{t_m}$ will have the same amplitude while pulses occurring almost 1s before $\overline{t_m}$ will have a slightly different information. In order to take this variability into account, we provide an exponentially decreasing weighting function to the pulses, i.e. we implement a boxcar integrator (BI) where each acquisition is made by a CDF.
This solution corresponds to subtracting the output of two BIs, one that is applied where the signal is present and the other one that is applied when only

the noise is present. Each integration window of the BI applied on the signal has a duration of T_{G1} while the other integration window has a duration of T_{G2}.
Following the same approach used in point C) to size the CDF, T_{G2} is larger than T_{G1} to avoid white noise doubling. Therefore, we must use two different time constants for the two integration windows, to ensure a proper CDF-like weighting function to the entire time interval of interest. In other words, if we used the same time constant for both integration windows, the weighting function applied to the noise would go to zero much sooner due to the fact that it discharges for a longer time interval during each period with respect to the signal one. Both weighting functions must go to zero within $\Delta T_{max} = 1s$, to avoid mixing up pulses carrying different information.
Since the BI applied to the signal discharges only for a time interval T_{G1} during each period of duration T_R, while the BI applied to the noise discharges for a time interval T_{G2} every T_R, we accordingly size the two time constants as follows:

$$\tau_1 = \frac{\Delta T_{max}}{5} * \frac{T_{G1}}{T_R}$$
$$\tau_2 = \frac{\Delta T_{max}}{5} * \frac{T_{G2}}{T_R}$$

Considering an amplitude of the two integration windows of $\frac{1}{T_{G1}}$ and $\frac{1}{T_{G2}}$, we can compute the signal and the white noise contributions as follows:

$$S_{BI+CDF} = \sum_{k=0}^{N} V_P\, e^{\frac{-kT_{G1}}{\tau_1}} \cong V_P \frac{\tau_1}{T_{G1}}$$

$$N^2_{w,BI+CDF} = \sum_{k=0}^{N} \frac{S_V}{2} \left(\frac{1}{T_{G1}} e^{\frac{-2kT_{G1}}{\tau_1}} + \frac{1}{T_{G2}} e^{\frac{-2kT_{G2}}{\tau_2}}\right) \cong \frac{S_V}{2}\left(\frac{1}{T_{G1}}\frac{\tau_1}{2T_{G1}} + \frac{1}{T_{G2}}\frac{\tau_2}{2T_{G2}}\right) =$$

$$= \frac{S_V}{4}\left(\frac{\tau_1}{T_{G1}^2} + \frac{\tau_2}{T_{G2}^2}\right)$$

Substituting the expressions of the two time constants defined above, we obtain:

$$S_{BI+CDF} = V_P \frac{1}{T_{G1}} * \frac{\Delta T_{max}}{5} * \frac{T_{G1}}{T_R} = V_P \frac{\Delta T_{max}}{5T_R}$$
$$N^2_{w,BI+CDF} = \frac{S_V}{4}\frac{\Delta T_{max}}{5T_R}\left(\frac{1}{T_{G1}} + \frac{1}{T_{G2}}\right)$$

It is worth stressing that the sizing of T_{G2} has been already determined based on the need to provide a high-pass filtering action. Thus, this is not a parameter that can be varied to maximize the signal-to-white-noise ratio.

The last contribution that needs to be computed is the one of 1/f noise. Since the boxcar is a low-pass filter but this noise is correlated, we can roughly estimate its contribution considering it as an offset in the time span of the boxcar weighting function duration. This means that the signal-to-1/f-noise ratio does not improve adding the BI to the CDF. We can use this approach to calculate the 1/f equivalent contribution. We will start from the signal-to-1/f-noise ratio of the CDF calculated in point C), that is:

$$(\frac{S}{N})_{1/f,CDF} = \frac{S_{CDF}}{\sigma_{1/f,CDF}} = \frac{\frac{2}{3}V_P}{\sigma_{1/f,CDF}}$$

By adding the BI, the amount of collected signal is increased by a factor α that can be simply derived as follows:

$$\alpha = \frac{S_{BI+CDF}}{S_{CDF}} = \frac{V_P \frac{\Delta T_{max}}{5T_R}}{\frac{2}{3}V_P} = \frac{3}{2}\frac{\Delta T_{max}}{5T_R}$$

Considering (at first approximation) the signal-to-1/f-noise to remain unchanged when the BI is added to the CDF can be mathematically expressed as follows:

$$(\frac{S}{N})_{1/f,BI+CDF} \approx (\frac{S}{N})_{1/f,CDF} = \frac{\alpha S_{CDF}}{\sqrt{\alpha^2 \sigma_{1/f,CDF}{}^2}}$$

The equivalent of 1/f noise contribution is:

$$\sigma_{\frac{1}{f},BI+CDF,eq} \approx \sqrt{\alpha^2 \sigma_{1/f,CDF}{}^2}$$

Such equivalent value can be quadratically summed to the white noise contribution to estimate the overall noise contribution and thus the obtainable SNR of the designed filtering scheme.

$$(\frac{S}{N})_{BI+CDF} \approx \frac{S_{BI+CDF}}{\sqrt{\sigma_{\frac{1}{f},BI+CDF,eq}{}^2 + \sigma_{w,BI+CDF}^2}} = \frac{S_{BI+CDF}}{\sqrt{\alpha^2 \sigma_{1/f,CDF}{}^2 + \sigma_{w,BI+CDF}^2}}$$

Considering the same sizing of point C), we have

$$T_{G,1} = \frac{2}{3}T_P$$

$$T_{G,2} = T_R - T_P.$$

$$\sigma_{1/f,CDF} \cong 1{,}7\mu V$$

and we can compute:

$$\alpha = \frac{3}{2}\frac{1s}{5 * 50\mu s} = 6000$$

and

$$\sigma_{w,BI+CDF} = \sqrt{\frac{S_V}{4}\frac{\Delta T_{max}}{5T_R}(\frac{1}{T_{G1}}+\frac{1}{T_{G2}})} = \frac{5nV}{\sqrt{Hz}} * \sqrt{4000 * \left(\frac{7}{40\mu s}\right)}$$
$$= 5\mu V * \sqrt{700} \cong 132\mu V.$$

Finally, we can write the SNR as follows:

$$(\frac{S}{N})_{BI+CDF} = \frac{\alpha S_{CDF}}{\sqrt{\alpha^2 {\sigma_{1/f,CDF}}^2 + {\sigma_{w,BI+CDF}}^2}} = \frac{6000 * \frac{2}{3}V_P}{\sqrt{(6000)^2(1{,}7\mu V)^2 + (132\mu V)^2}}$$

The low pass filtering action of the BI has made the white noise contribution negligible with respect to the 1/f noise one, which sets the ultimate limit to the sensitivity of the chosen filtering scheme.
In this case we have

$$V_{P,MIN} \approx \frac{3}{2}\sigma_{1/f,CDF} = \frac{3}{2}1{,}7\mu V = 2{,}55\mu V$$

Extra: for the sake of completeness we will now add a comment on the scenario of designing a solution to point D) in the conditions of point B), i.e. with random pulse interarrival time. This scenario is similar to the one with a fixed rate of pulses except for the fact that the random arrival time of point B) prevents the design and exploitation of a noise-only acquisition window at a fixed time with respect to the signal. From a high-level perspective, the approach is the same above discussed to solve point D) with a fixed repetition rate: multiple pulses should be collected applying a variable weight to rely more on the most recent information, and a baseline subtraction must be included in both cases. Nevertheless, the implementation of a practical solution in the two cases would be different because the randomness of the arrival time of pulses makes it more challenging to collect only the noise and avoid any pulse presence during such noise-only acquisition intervals. Computing a rough estimation of the obtainable result with random interarrival time would be extremely complex.

PROBLEM 3

June 22nd, 2021 – Pb. 1

In this problem, an exponential signal produced by a current generator is subject to an integration action provided by a parallel RC network. The same fate happens to the current noise contribution, while the voltage noise contribution experiences its peculiar transfer function. In this scenario, the optimum filter on a single pulse is discussed first, and then the exploitation of multiple pulses is considered. Finally, 1/f noise contribution is taken into account too.

As sketched in the figure below (Fig.1), a current signal is acquired by a preamplifier featuring a very high input impedance (of the order of 1 GΩ), a band limited by a single pole at a frequency f_P = 200 MHz and two input-referred noise generators with unilateral spectral densities $\sqrt{S_V} = 1nV/\sqrt{Hz}$ and $\sqrt{S_I} = 1pA/\sqrt{Hz}$. C_L=2pF and R_L=10MΩ represent the total capacity and resistance, respectively, between the sensor output and ground. The detector delivers trains of exponential pulses with unknown amplitude A_P, decay time constant T_P=20ns and repetition rate r_P=1kHz. The duration of the measurement can span from 1 to 20 min.

A) Describe in detail how you can calculate the optimum filter and calculate the minimum amplitude that could be detected for each single pulse.

B) Considering now that the amplitude of the pulses slowly changes with a timescale of 1s, design a suitable filter to exploit this new information and calculate the minimum detectable signal amplitude with the proposed solution.

C) Considering now that the current noise of the preamplifier has also a 1/f component with f_c = 50kHz, evaluate its effect on the measurement in the conditions of point B). Then provide a solution to limit its effect and calculate the minimum detectable signal amplitude with the proposed solution.

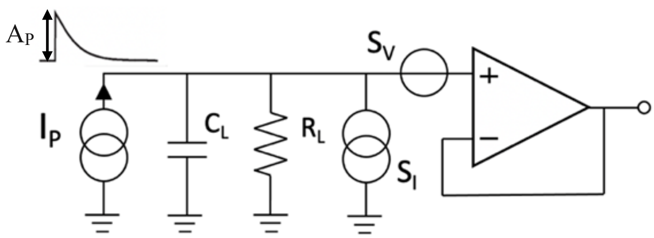


Fig.1

SOLUTION

The measurement scenario presented in this problem consists of a current generator producing multiple pulses with a fixed repetition rate (r_P=1kHz). The original shape of each pulse is fully characterized by its decay time constant T_P=20ns, while the unknown amplitude A_P carries the information to be extracted. Nonetheless, the shape of the signal is modified by the integration provided by the parallel resistive-capacitive network composed by C_L and R_L (the input resistance of the preamplifier is negligible being much higher than R_L). The impact of the RC network is due to the fact that its time constant τ_L= C_LR_L = 20μs is much lower than the signal time constant T_P=20ns. At the same time, the preamplifier that is used to extract the signal features two input-referred noise generators: a current generator that, being in parallel to the signal current generator, undergoes to the exact same integration process, and a voltage generator that is transferred to the output with a unitary gain up to the preamplifier single pole at f_P = 200 MHz.

A) *Describe in detail how you can calculate the optimum filter and calculate the minimum amplitude that could be detected for each single pulse.*

In this point we are asked to design the theoretically best filter for the scenario of interest and calculate the sensitivity that can be achieved on the single pulse. This means that the information of multiple pulses must not be mixed up.

Let's first of all focus on the single pulse and the associated noise. The optimum filter theory requires a perfectly-known shape of the signal and white noise. While the shape of the signal is known - at least at the input of the chain - the integration process that affects the current noise generator makes this contribution non-white at the voltage input of the preamp. On the other hand, the input network has no effect on the voltage noise contribution that remains unchanged at the input of the preamplifier. Overall, the total voltage noise at the input of the preamplifier is given by the following expression:

$$S_{TOT} = S_V + \frac{S_I R_L^2}{1 + \left(\frac{f}{f_L}\right)^2}$$

where $f_L = \frac{1}{2\pi R_L C_L}$ is the frequency of the pole given by RC network.

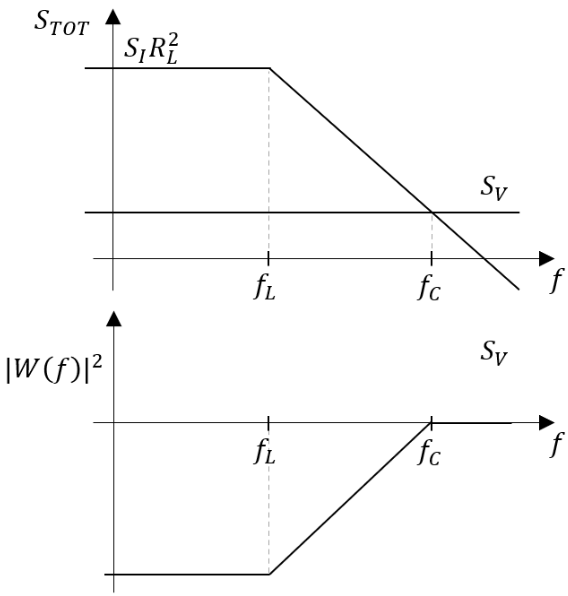
The integration process has introduced a $1/f^2$ component which prevents the immediate application of the matched filter theory. Nonetheless, this noise spectrum can be made white by designing an appropriate whitening filter. In order to whiten the overall noise spectrum graph, that is shown below, first of all we need to compensate the pole at f_L with a zero at the same frequency. Secondly, we can observe that, above the corner frequency f_C, the already-white contribution given by the voltage noise generator becomes dominant with respect to the current noise.

This frequency can be found using the properties of the Bode diagram plot. Having a slope of $-40dB/dec$ above f_L, we have:

$$S_I R_L^2 * f_L^2 = S_V * f_C^2$$

$$f_C = f_L \sqrt{\frac{S_V}{S_I R_L^2}} = 80MHz.$$

Above such frequency we can simply provide a unitary constant gain to keep the noise white. The Bode plot of the necessary whitening filter is shown below the noise spectrum.



The designed whitening filter produces a constant noise spectrum equal to S_V.

Now we have to focus on the signal. The introduction of the whitening filter in the acquisition chain affects also the signal, that is therefore subject to a double filtering action: the integration due to the RC network and the one given by the transfer function of the whitening filter. Nonetheless, the zero of the whitening filter coincides with the pole of the RC stage, thus leaving only a single pole at f_C on the signal path. In other terms, the overall transfer function of the signal is a low-pass filter with a single pole at f_C.

The time constant of this integration given by the pole $f_C = 80MHz$ is $\tau_{LPF} = \frac{1}{2\pi 80MHz} = 2ns$ which is one-order-of-magnitude lower than the decay time constant $T_P = 20ns$. This means that the low pass filter action doesn't affect the signal shape. The current signal is thus simply converted into a voltage by a multiplication for the low-frequency gain. We can calculate such gain by considering the DC transfer function of the RC network, that is equal to R_L, and the DC gain of the whitening filter. Again, by using the properties of the Bode diagram, we can calculate that the DC gain that the whitening filter applies to the signal is equal to

$$|W(f)|_{DC} = \frac{f_L}{f_C} = 10^{-4}$$

Overall, at the output of the designed whitening filter we have a white noise equal to S_V and a voltage signal described by the following mathematical expression:

$$v_s(t) = A_P R_L \frac{f_L}{f_C} e^{\frac{-t}{T_P}}$$

where all parameters are known except for original signal amplitude A_P.
In this scenario, finally having a white noise spectrum and a known signal amplitude we can complete the design of the optimum filter with the matched filter. To this aim, the signal waveform can be divided into $A = A_P R_L \frac{f_L}{f_C}$ and b(t)= $e^{\frac{-t}{T_P}}$.

$$(\frac{S}{N})_{OPT} = \frac{A}{\sqrt{S_B}}\sqrt{k_{bb}(0)} = \frac{A}{\sqrt{\frac{S_V}{2}}}\sqrt{\frac{T_P}{2}} = \frac{A_P R_L \frac{f_L}{f_C}}{\sqrt{\frac{S_V}{2}}}\sqrt{\frac{T_P}{2}}$$

Finally, we compute the sensititvity that could be achieved with the optimum filter:

$$A_{P,MIN} = \frac{\sqrt{\frac{S_V}{T_P}}}{R_L \frac{f_L}{f_C}} = \frac{\frac{1nV}{\sqrt{Hz}\sqrt{20ns}}}{1k\Omega} \cong 7{,}07nA$$

Comment: the dominant noise contribution is given by the current noise that gets integrated by the RC input network. The whitening filter restores the original shape of the current noise. Since the signal has the same nature of such noise contribution and it goes through the same filtering stages, also its original shape is restored by the whitening filter.

B) *Considering now that the amplitude of the pulses slowly changes with a timescale of 1s, design a suitable filter to exploit this new information and calculate the minimum detectable signal amplitude with the proposed solution.*

The possibility of exploiting multiple pulses carrying similar information can significantly improve the signal-to-noise ratio. Since the amplitude of the pulses slowly changes over time, the most recent information should be weighted more than the older one. To this aim, we can use a Boxcar Integrator (BI) or a Ratemeter Integrator (RI). In this problem the number of pulses that can be summed together is limited by the timescale of the pulse amplitude and the repetition rate is fixed. In this scenario, using a RI or a BI leads to the exact same result. Before going into the details of one of these filters, it is worth highlighting that any solution exploiting multiple pulses to improve the SNR requires a sync signal to locate the single pulse that is buried into noise. Therefore, we will assume that a sync signal is available.

Comment*: in a real case scenario like the one presented in problem, it is convenient to compute the result that could be achieved if a sync signal was available and, only in case it is proven effective to meet the application requirements, the designer will make the effort of finding or deriving a suitable sync signal.*

This point of the problem asks to design a suitable filter for multiple pulses. Given the exploitation of a BI or a RI, some filtering action must be also provided on the single pulse in order to maximize the SNR. First of all, a high pass filtering action is necessary to filter out the integrated current noise contribution. To this aim, the whitening filter that has been already designed in point A) is maintained as it fully preserves the signal and produces a white noise that can be limited with a low pass filter. Among low-pass filters, a practical gated integrator (GI) can be designed considering that a sync signal must be available as previously discussed. It can be demonstrated that the best sizing for the duration T_G of an integration window applied to an exponential signal coming with white noise is $T_G = 1.25\tau$, where τ is the decay time of the signal. In this case, we will thus consider a GI having A=1 (as it does not affect the SNR) and $T_G = 1.25T_P$.
Recalling that the noise spectrum at the output of the whitening filter is flat and equal to S_V while the signal is $v_s(t) = A_P R_L \frac{f_L}{f_C} e^{\frac{-t}{T_P}}$, we can compute the signal to noise ratio of the GI applied to one pulse, that is:

$$\left(\frac{S}{N}\right)_{GI} = \frac{A_P R_L \frac{f_L}{f_C} T_P \left(1 - e^{-\frac{T_G}{T_P}}\right)}{\sqrt{\frac{S_V}{2} T_G}} = \frac{A_P R_L \frac{f_L}{f_C} T_P (1 - e^{-1,25})}{\sqrt{\frac{S_V}{2} 1,25 T_P}}$$

With the optimized GI applied to the single pulse we obtain:

$$A_{P,MIN} = \frac{\sqrt{\frac{S_V}{2} 1,25 T_P}}{R_L \frac{f_L}{f_C} T_P (1 - e^{-1,25})} \cong 7,85 nA$$

As expected, the sensitivity that is obtained with the practical gated integrator is worse than the theoretically best one computed in point A) with the optimum filter. This result can be now improved by exploiting either a BI or a RI. For this computation we will consider a RI. The design of this filter requires to size its time

constant to ensure that the weighting function covers only pulses having a similar amplitude. In this case, we can have a weighting function duration of 1s by sizing the time constant $\tau_{RI} = \frac{1s}{5} = 200ms$. The amplitude of the RI weighting function can be chosen equal to 1 as in any case it does not have any impact on the SNR.
In this scenario, the exponential averaging applied to many integration windows enhances by a factor $\alpha_{S,RI} = \tau_{RI} r_P$ with respect to the single GI, where r_P is the repetition rate of pulses. Concerning noise, the exponential averaging boosts the noise by a factor $\alpha_{N,RI} = \sqrt{\frac{\tau_{RI} r_P}{2}}$ with respect to the single GI. In the end, adding the designed RI to the GI provides a SNR enhancement by a factor $IF_{RI} = \frac{\alpha_{S,RI}}{\alpha_{N,RI}} = \sqrt{2\tau_{RI} r_P}$. Therefore, the sensitivity of the system is improved by such enhancement factor:

$$A_{P,MIN,GI+RI} = \frac{A_{P,MIN,GI}}{IF_{RI}} = \frac{7{,}85nA}{\sqrt{2 * 200ms * 1kHz}} \cong 0{,}39nA$$

Extra: for the sake of completeness, we will here report the sizing of the BI and the relative computation of the improvement factor with respect to the GI highlighting that in the given scenario the BI and the RI provide the exact same result. As for the RI, the overall duration of the BI weighting function must be equal to 1s. However, it must be taken into account that the weighting function of the BI only decreases during the signal acquisition time intervals, i.e. for a time interval T_G for each repetition period of duration $T_R = 1/r_P$. Therefore, the time constant of the BI must be reduced by a factor of $\frac{T_G}{T_R}$ leading to the following expression:

$$\tau_{BI} = \frac{1s}{5} * \frac{T_G}{T_R}$$

In this case, the effect of the exponential averaging (considering a BI weighting function amplitude equal to 1) is $\alpha_{S,BI} = \frac{\tau_{BI}}{T_G}$ for the signal and $\alpha_{N,RI} = \frac{\tau_{BI}}{2T_G}$. The resulting improvement factor of the BI is $IF_{BI} = \frac{\alpha_{S,BI}}{\alpha_{N,BI}} = \sqrt{\frac{2\tau_{BI}}{T_G}}$. By substituting the expression of the BI time constant, we obtain:

$$IF_{BI} = \sqrt{\frac{2\tau_{BI}}{T_G}} = \sqrt{\frac{2\frac{1s}{5} * \frac{T_G}{T_R}}{T_G}} = \sqrt{2\frac{1s}{5}\frac{1}{T_R}}$$

which highlights that IF_{BI} is exactly the same as IF_{RI}.

C) *Considering now that the current noise of the preamplifier has also a 1/f component with fc = 50kHz, evaluate its effect on the measurement in the conditions of point B). Then provide a solution to limit its effect and calculate the minimum detectable signal amplitude with the proposed solution.*

First of all, we need to understand the effect of an additional 1/f noise component on the overall noise spectrum. The integration provided by the RC network to current noise causes the additional contribution to produce a $1/f^3$ component. For this reason, and for all the other reasons already listed in point B), we must keep the whitening filter. The effect of this filter on the 1/f noise component can be simply evaluated considering that it was designed in point A) to restore the original shape of the current noise contribution. In point A) the noise was white and it is made white again by the whitening filter that nullifies the action of the RC network. Analogously, in this point at the output of the whitening filter we obtain both a white noise and a 1/f noise component with the given noise frequency f_C = 50kHz, since this is well below the pole of the whitening filter (where the only-white voltage noise component becomes the dominant contribution). In this scenario, an additional high-pass filtering action is necessary. Since we are in the scenario of point B) where gated integrators have been already used, we can resort to correlated double filtering to provide a high-pass filtering action only on the noise, thus avoiding any reduction of the signal. Moreover, this approach is preferable to constant-parameter filters like the CR high-pass filter due to the repetitive nature of the signal which could have set some limitations in the design of a CR network.

Since the signal is only positive in this problem, any correlated double filtering acquisition scheme requires a time interval where only noise is present in order to avoid any impact on the signal. There are basically two different options:

- single filtered acquisition of the noise before starting the measurement, i.e. a zero setting (ZS)
- repeated noise acquisition during each period when the signal is not present, that is a correlated double filtering (CDF).

The ZS has the advantage of being simpler in terms of implementation and it is typically already provided by many instruments like oscilloscopes; on the other hand, the CDF would provide a better filtering to 1/f noise thanks to the shorter time distance between integration windows with respect to the ZS. We will now design a ZS to estimate the sensitivity that can be achieved with a simple solution. In a real application this would be the first step to understand if a more complex solution is necessary. The ZS will be added to the acquisition chain designed in point B), consisting of the whitening filter and the RI with optimized GI on each pulse ($T_G = 1{,}25T_P$). The duration of the integration window T_W of the ZS can be as long as desired as this acquisition occurs only once before the signal starts. To ensure no significant contribution to the white noise, we can have $T_W \gg T_G$. It is worth saying that this choice does not have any significant effect on the 1/f contribution since the time distance between the integration windows is orders of magnitude higher than the integration window duration. Now we need to compute the signal and the white and the 1/f noise contributions. We will first consider the ZS composed by one GI

on a single pulse and the integration window that precedes the measurement, and then we'll discuss how to take into account the RI effect.
Concerning the ZS, we'll consider a integration window on the signal featuring $T_G = 1{,}25T_P$ and $A = 1/T_G$ while the integration windows on noise features a duration $T_W = 100T_G$ and $A = 1/T_W$. In this way, both integration windows have a unitary area and the 1/f noise can be computed by means of the usual formula. 1/f noise computation depends on the time distance between the two integration windows: in the worst case scenario, that is the acquisition of a pulse at the end of a long measurement, such distance is $T_{MEAS,max} = 20$ minutes. Now we can compute the three equivalent bandwidths for noise calculation:

- GI, $f_{LPF1} = \frac{1}{2T_G} = 20MHz$;
- Noise acquisition, $f_{LPF2} = \frac{1}{2T_W} = 200kHz$
- ZS, $f_{HPF,eq} \cong \frac{1}{2\pi T_{MEAS,max}} = 132{,}5\mu Hz$

The noise of a ZS applied to the last pulse of a measurement of duration $\mathrm{T_{MEAS,max}}$ is given by the two following contributions:

$$\sigma_{w,ZS} \cong \sqrt{S_V * (\mathrm{f_{LPF1}} - \mathrm{f_{HPF,eq}}) + S_V * (\mathrm{f_{LPF2}} - \mathrm{f_{HPF,eq}})} \cong \sqrt{S_V * \mathrm{f_{LPF1}}} \cong 4{,}47\mu\mathrm{V}$$

$$\sigma_{\frac{1}{f},ZS,max} \cong \sqrt{S_V f_C[\ln\left(\frac{\mathrm{f_{LPF1}}}{\mathrm{f_{HPF,eq}}}\right) + \ln\left(\frac{\mathrm{f_{LPF2}}}{\mathrm{f_{HPF,eq}}}\right)]} \cong 1{,}53\mu\mathrm{V}$$

The signal is the same computed in point B) with only the GI but scaled by $\frac{1}{T_G}$:

$$S_{ZS} = A_P R_L \frac{f_L}{f_C} T_P(1 - e^{-1{,}25}) * \frac{1}{T_G} = A_P R_L \frac{f_L}{f_C}\frac{1}{1{,}25}(1 - e^{-1{,}25})$$

The last step of this computation involves the effect of the ratemeter integrator on the pulse. The effect of the RI on the signal and on the white noise has been already computed in point B) and it is reported here for the sake of clarity:

$$S_{ZS+RI} = \alpha_{S,RI} * S_{ZS} = \tau_{RI} r_P * A_P R_L \frac{f_L}{f_C}\frac{1}{1{,}25}(1 - e^{-1{,}25})$$

$$\sigma_{w,ZS+RI} = \alpha_{N,w,RI} * \sigma_{w,ZS} \cong \sqrt{\frac{\tau_{RI} r_P}{2}} * \sqrt{S_V * \mathrm{f_{LPF1}}} \cong 44{,}7\mu V$$

The computation of the final SNR requires to calculate the 1/f noise contribution. To this aim, two main aspects have to be taken into account. First, the 1/f noise contribution collected on each pulse is different since the distance of each acquisition with respect to the initial noise integration window is different. Such distance increases with time, so the highest contribution is collected after 20 minutes, that is the maximum duration of the measurement ($T_{MEAS,max}$). Secondly, the RI provides a low pass filtering action which does not have a major impact in limiting the 1/f noise contribution. Since the exact computation of the 1/f noise contribution in this scenario is too complex, we must resort to some approximation in order to estimate such contribution. To be conservative, we can consider a constant 1/f noise contribution in all acquisitions with a value that corresponds to the worst case scenario, i.e. the 1/f noise acquired with the ZS having a time distance between the integration windows as high as $T_{MEAS,max}$. Considering this noise to be constant over all the acquisitions means that the signal to 1/f noise ratio does not improve with the exploitation of the RI. This can be translated into formulas by applying the same gain of the signal, $\alpha_{S,RI}$, to 1/f noise:

$$\sigma_{1/f,ZS+RI} \approx \alpha_{S,RI} * \sigma_{\frac{1}{f},ZS,max} = \sqrt{(\tau_{RI} r_P)^2 S_V f_C [ln\left(\frac{f_{LPF1}}{f_{HPF,eq}}\right) + ln\left(\frac{f_{LPF2}}{f_{HPF,eq}}\right)]} = 306\mu V$$

Now it is possible to estimate the overall noise contribution by quadratically summing up the white and the 1/f noise contributions:

$$\sigma_{TOT,ZS+RI} = \sqrt{\sigma_{w,ZS+RI}{}^2 + \sigma_{1/f,ZS+RI}{}^2} \approx 309\mu V$$

mainly due to 1/f noise.

Finally, we can estimate the sensitivity of this system:

$$A_{P,MIN,ZS+RI} \approx \frac{\sigma_{TOT,ZS+RI}}{\tau_{RI} r_P * R_L \frac{f_L}{f_C} \frac{1}{1{,}25} (1 - e^{-1{,}25})} = 2{,}71nA.$$

PROBLEM 4

January 17th, 2024 – Pb. 1

The focus of this problem is on digital filtering. An exponential signal is read-out by an amplifier affected by white noise and the fed to a digital acquisition chain consisting of a sampler and a digital elaboration unit. Two different scenarios in terms of sampling frequency and noise correlation time are discussed.

We want to measure the amplitude V_P of an exponential signal (as shown in the figure below) featuring a decay time of 100ns, coming from the output of an amplifier limited by a single pole ($f_A = 100MHz$) and affected by an input referred white noise with unilateral spectral density $\sqrt{S_V} = \frac{1nV}{\sqrt{Hz}}$. The signal is acquired by means of a digital acquisition chain consisting of a sampler and a programmable elaboration unit. A sync signal is available.

A) Considering a sampler with a maximum sampling frequency of 50MHz, discuss and design a suitable digital filtering scheme and calculate the corresponding minimum signal amplitude that could be measured with the proposed solution.

B) Assuming now that an additional wideband noise component comes with the signal (low-frequency unilateral spectral density $\sqrt{S_{V,NS}} = \frac{100nV}{\sqrt{Hz}}$, one pole at 1MHz, lorentzian spectrum) and that the maximum sampling frequency is 5MHz, discuss and design a digital solution for this case and calculate the new minimum measurable signal.

C) Demonstrate in detail, from a theoretical point of view, what the optimal analog filter would be for the scenario of point A) and comment on the result.

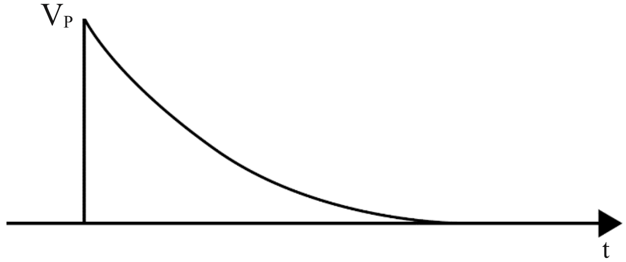


Fig.1

SOLUTION

The signal of interest in this problem is an exponential signal which can be mathematically described as follows:

$$s(t) = \begin{cases} V_P e^{-\frac{t}{\tau_P}} & for\ t \geq 0 \\ 0 & otherwise \end{cases}.$$

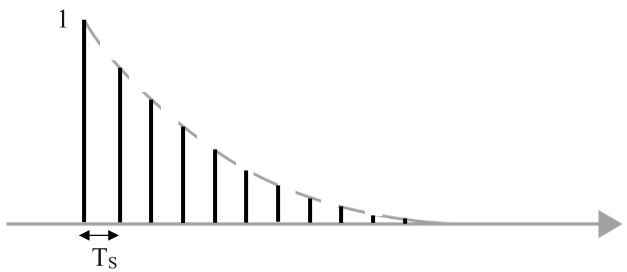
where the decay time constant τ_P is equal to 100ns. The analog signal is acquired by an amplifier featuring a bandwidth limited by a single pole at $f_A = 100MHz$. The Fourier transform of the signal is

$$S(f) = \frac{V_P \tau_P}{1 + j\omega f \tau_P}$$

meaning that most of its frequency content is within $f_P = \frac{1}{2\pi\tau_P} = 1.6MHz$. Since f_A is much bigger than f_P, the signal shape is not changed by the preamplifier. Concerning noise, the amplifier introduces a wideband noise component featuring a lorentzian spectrum that is the result of the amplifier filtering action itself on the white noise that is present at its input.

A) *Considering a sampler with a maximum sampling frequency of 50MHz, discuss and design a suitable digital filtering scheme and calculate the corresponding minimum signal amplitude that could be measured with the proposed solution.*

The signal features a known shape and it comes with a wideband noise. Moreover, a sync signal allows us to precisely know when the signal is present. For all these reasons, the acquisition can be optimized following an approach that derives from the matched filter theory. In this case, the digital filtering scheme can be based on the acquisition of as many samples as possible with the given sampling frequency and within $5\tau_P$ and each sample will be given a weight that is proportional to the signal shape. The resulting weighting function can be sketched as follows:



where T_S is the sampling time. The weighting function is mathematically described by the following formula:

$$w(k) = 1e^{-\frac{kT_S}{\tau_P}} \quad for\ k \in [0, +\infty)$$

Before computing the SNR it is necessary to evaluate the potential correlation among noise samples. As discussed above, at the output of the amplifier the noise features a Lorentzian spectrum limited by $f_A = 100MHz$. Therefore, its correlation function is a double exponential with $\tau_{corr} = \frac{1}{2\pi f_A} = 1.6ns$. Since $T_S = \frac{1}{f_{S,MAX}} = \frac{1}{50MHz} = 20ns$ is higher than $5\tau_{corr} = 8ns$, noise samples are uncorrelated. Therefore, the SNR can be computed as follows:

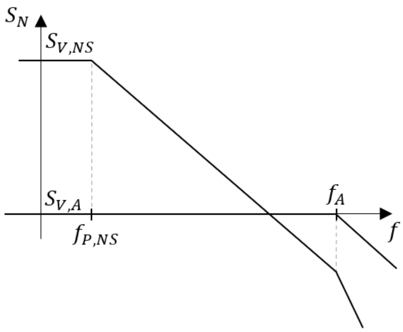
$$\frac{S}{N} = \frac{\sum_{k=0}^{+\infty} V_P e^{-\frac{kT_S}{\tau_P}} * 1e^{-\frac{kT_S}{\tau_P}}}{\sqrt{\sum_{k=0}^{+\infty} S_V \frac{\pi}{2} f_A * 1^2 e^{-\frac{2kT_S}{\tau_P}}}} = \frac{\sum_{k=0}^{+\infty} V_P e^{-\frac{2kT_S}{\tau_P}}}{\sqrt{\sum_{k=0}^{+\infty} S_V \frac{\pi}{2} f_A * 1^2 e^{-\frac{2kT_S}{\tau_P}}}} = \frac{V_P \frac{\tau_p}{2T_S}}{\sqrt{S_V \frac{\pi}{2} f_A \frac{\tau_p}{2T_S}}} = \frac{V_P}{\sqrt{S_V \frac{\pi}{2} f_A}} \sqrt{\frac{\tau_p}{2T_S}}.$$

The minimum amplitude of the signal that can be measured with the proposed solution is:

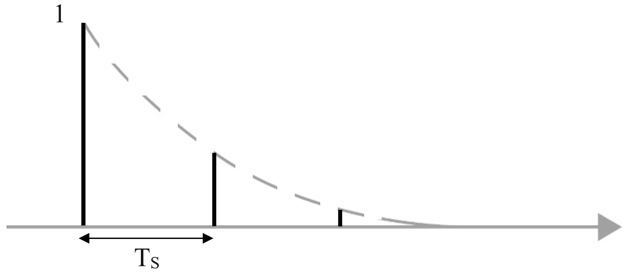
$$V_{P,MIN} = \sqrt{S_V \frac{\pi}{2} f_A} * \sqrt{\frac{2T_s}{\tau_p}} = 12.5\mu V \sqrt{\frac{2 * 20ns}{100ns}} = 7.9\mu V$$

B) *Assuming now that an additional wideband noise component comes with the signal (low-frequency unilateral spectral density $\sqrt{S_{V,NS}} = \frac{100nV}{\sqrt{Hz}}$, one pole at 1MHz, lorentzian spectrum) and that the maximum sampling frequency is 5MHz, discuss and design a digital solution for this case and calculate the new minimum measurable signal.*

In this point two key aspects of the acquisition scenario have changed with respect to point A). First of all, an additional noise component comes with the signal in this point, and it adds directly to the noise of the amplifier. Such component is much higher than the amplifier noise at low frequencies and it drops after $f_{P,NS}$=1MHz, while both components are limited by the amplifier pole as sketched in the following graph:



Secondly, the maximum sampling frequency in this point is as high as 5MHz, which corresponds to a minimum sampling time of 200ns. In this scenario, only 3 samples of the signal (i.e. within $5\tau_P$) can be acquired, as sketched in the following figure:



It is worth noting that also in this case it is convenient to provide a weight that is proportional to the signal shape in order to maximixe the SNR.
Concerning the noise, the dominant component coming with the signal can be approximated to a single-pole limited spectrum, i.e. the second pole given by f_A can be neglected. This approach is not only reasonable since f_A is much bigger than $f_{p,NS}$, but it is also conservative. By doing so, we can approximate the spectrum of this component with a lorentzian one featuring $\tau_{corr,NS} = \frac{1}{2\pi f_{P,NS}} = 160ns$. As a result, we can derive that noise samples acquired with the sampler operating at its maximum frequency of 5MHz are correlated. Therefore, in order to compute the overall noise acquired with the design weighting function it is necessary to compute the autocorrelation function $k_{ww}(\gamma)$. This function is symmetrical and described by the following terms:

$$k_w(0) = 1^2 + (e^{-\frac{T_S}{\tau_P}})^2 + (e^{-\frac{2T_S}{\tau_P}})^2 = 1 + e^{-4} + e^{-8} \cong 1.02.$$

$$k_w(\mp T_S) = 1 * (e^{-\frac{T_S}{\tau_P}}) + (e^{-\frac{T_S}{\tau_P}}) * (e^{-\frac{2T_S}{\tau_P}})^2 = e^{-2} + e^{-6} \cong 0.14.$$

$$k_w(\mp 2T_S) = 1 * (e^{-\frac{2T_S}{\tau_P}}) = e^{-4} \cong 0.02.$$

Such autocorrelation function must be multiplied by the autocorrelation function of the correlated noise to compute its contribution. As previously discussed, we can describe the spectrum of this noise component with a Lorentzian shape featuring $\sigma_{NS}{}^2(0) \cong \sqrt{S_{V,NS}\frac{\pi}{2}f_{P,NS}} = 125\mu V$ and an exponential shape with time constant $\tau_{corr,NS} = 160ns$. Now the contribution of this noise source can be computed as follows:

$$\sigma_{NS,OUT}{}^2 = \sigma_{NS}{}^2(0)[k_w(0) + 2 * e^{-\frac{T_S}{\tau_{corr.NS}}} * k_w(\mp T_S) + 2 * e^{-\frac{2T_S}{\tau_{corr.NS}}} * k_w(\mp 2T_S)].$$

Therefore, the signal to noise ratio has the following expression:

$$\frac{S}{N} \cong \frac{V_P\left(1+e^{-\frac{T_S}{\tau_P}}+e^{-\frac{2T_S}{\tau_P}}\right)}{\sigma_{NS,OUT}} \cong \frac{1.15V_P}{\sigma_{NS}(0)*\sqrt{1.02+0.08+0.003}} \cong \frac{1.15V_P}{131.3\mu V}$$

With the designed solution, the minimum amplitude of the signal that can be measured is

$$V_{P,MIN} = \frac{131.3\mu V}{1.15} \cong 114.2\mu V$$

Extra: for the sake of completeness, we will here report the computation of the noise contribution given by the noise of the amplifier, showing that it can be neglected with respect to the other dominant component. The noise of the amplifier is only limited by the pole of the amplifier itself, providing a correlation time constant as low as 1.6ns as computed in point A). For this reason, noise samples of this source are uncorrelated and their contribution can be computed as follows:

$$\sigma_{N,A} = \sqrt{S_V\frac{\pi}{2}f_A * \left(\left(1+e^{-\frac{2T_S}{\tau_P}}+e^{-\frac{4T_S}{\tau_P}}\right)\right)} = 12.6\mu V.$$

The two noise components would sum up quadratically, leading to an overall noise:

$$\sigma_{N,TOT} = \sqrt{{\sigma_{N,A}}^2 + {\sigma_{NS,OUT}}^2} = 114.9\ \mu V$$

which is extremely close to the value computed above considering only the dominant noise contribution.

C) *Demonstrate in detail, from a theoretical point of view, what the optimal analog filter would be for the scenario of point a) and comment on the result.*

In this point a theoretical demonstration is required which is not reported here since it is out of the scope of this exercise book. Just to guide the reader, it can be rapidly observed that in point A) a signal of a known shape is accompanied by only white noise. Therefore, the answer to this question can be found in the matched filter theory.

PROBLEM 5

February 2nd, 2024 – Pb. 1

This problem addresses the case of a signal that comes with an undesired superimposed baseline and accompanied by white noise. Thanks to the differences between the signal of interest and the baseline and noise, it will be shown how it is possible to maximize the collection of the signal while getting rid of most of the undesired elements both with a digital and an analog approach.

We want to measure the amplitude of a sinusoidal signal ($f_S = 100Hz$) that is superimposed to a non-negligible undesirable baseline that slowly varies on a time scale in the order of tens of seconds. The readout circuit consists of an amplifier featuring a single pole ($f_A = 100MHz$) and affected by an input referred white noise with unilateral spectral density $\sqrt{S_V} = \frac{10\text{nV}}{\sqrt{\text{Hz}}}$. The frequency of the signal can be easily derived from an auxiliary synchronized sinusoidal reference that is available with a high SNR.

A) Assuming that you can only carry out digital filtering with a maximum sampling frequency of 200Hz, and that the amplitude of the signal varies on a timescale in the order of a second, design a filter that allows you to extract the desired signal amplitude with high sensitivity. Evaluate the minimum signal amplitude that could be measured with the designed solution.

B) Being now able to carry out any kind of filtering, discuss and evaluate how the answer to the previous point changes.

C) Taking a generic gated integrator as an example, describe in detail (from a theoretical point of view) in the time domain the difference between the signal-to-noise ratio obtainable with analogue and digital filtering as the sampling frequency varies.

SOLUTION

A signal of interest featuring a known sinusoidal shape comes superimposed to an undesired baseline. These two elements substantially differ on their timescale of evolution. Indeed, the signal has a repetition period of 10ms, while the waveform of the baseline, that is not exactly known, is said to be varying on a timescale in the order of tens of seconds. This difference can be the key to separating the signal of interest from the baseline. Both elements are collected by the preamplifier which adds a wideband (ideally white at the input) noise component. Finally, the text states that the frequency of the signal can be easily derived from an auxiliary reference signal that is available with a high signal to noise ratio. By exploiting such reference signal it is possible to avoid any detrimental effect potentially produced by a slight difference between the nominal and the real signal frequency.

A) *Assuming that you can only carry out digital filtering with a maximum sampling frequency of 200Hz, and that the amplitude of the signal varies on a timescale in the order of a second, design a filter that allows you to extract the desired signal amplitude with high sensitivity. Evaluate the minimum signal amplitude that could be measured with the designed solution.*

In this point a digital acquisition scheme is provided with a maximum sampling frequency of 200Hz, that is twice the signal frequency. In other terms, at most two samples of the signal can be acquired for each period. First of all, it can be observed that the sampler follows a preamplifier featuring a single pole at 100MHz. Since the preamplifier pole frequency is orders of magnitude higher than the signal frequency, the signal is not affected at all by such frequency limitation. Analogously, the baseline that varies on a time scale even lower than the signal is not affected by the preamplifier filtering action. The only effect of the preamplifier pole is on noise, which is said to be white at the input of the preamp, thus it features a Lorentzian spectrum at the output of the preamplifier itself.
Concerning the signal, its amplitude is said to be varying on a timescale in the order of a second. This means that multiple periods within this time frame carry similar information, thus providing an opportunity to collect more information than just two samples on a single period. In order to achieve the requested high sensitivity, it is necessary to get rid of the undesired baseline. To this aim, a high pass filtering action must be provided.
Exploiting the features of the signal, a correlated double sampling (CDS) can be applied to multiple periods as follows: first of all, two samples within the signal period are acquired corresponding to the signal peak and its minimum. The digital subtraction of these two samples has a twofold advantage: it doubles the signal and it eliminates the undesired baseline. Secondly, multiple CDS like the one just described are applied to adjacent periods for an overall duration of 1s. The digital filter will provide an exponentially-decreasing weighting to multiple periods, where the maximum (unitary) weight is applied to the most recent information.
The signal to noise ratio is computed in two steps: first, the SNR of a single CDS is computed; then, an exponential weighting of multiple CDSs is considered.
The signal and noise resulting from a CDS are, respectively:

$$S_{CDS} = 2V_P$$

$$N_{CDS} = \sqrt{2S_V \frac{\pi}{2} f_A}$$

The exponential weighting is sized considering an overall duration of the weighting function of 1s, thus requiring

$$\tau_F = \frac{1s}{5}$$

The resulting signal and noise acquired by means of multiple exponentially-weighted CDSs is:

$$S_{TOT} = \sum_{k=0}^{+\infty} S_{CDS} e^{-\frac{kT_S}{\tau_F}} = 2V_P * \frac{\tau_F}{T_S}$$

$$N_{TOT} = \sum_{k=0}^{+\infty} N_{CDS} e^{-\frac{2kT_S}{\tau_F}} = \sqrt{2S_V \frac{\pi}{2} f_A * \frac{\tau_F}{2T_S}}$$

where $T_S = 1/f_S$.

The resulting signal to noise ratio is:

$$\frac{S}{N} = \frac{2V_P \sqrt{\frac{\tau_F}{T_S}}}{\sqrt{S_V \frac{\pi}{2} f_A}}$$

With such acquisition scheme, the minimum amplitude that can be measured is:

$$V_{P,MIN} = \frac{1}{2} \sqrt{S_V \frac{\pi}{2} f_A * \frac{T_S}{\tau_F}} = 28\mu V$$

B) *Being now able to carry out any kind of filtering, discuss and evaluate how the answer to the previous point changes.*

The freedom given in this point to design the filtering scheme should be used to achieve the best possible sensitivity. To this aim, it is crucial to highlight that the signal shape is known, except for the exact variation of the amplitude of the sinusoidal signal over time. This signal comes with wideband noise, thus allowing us to exploit the optimum filter theory. Following this approach, the first stage of the filtering scheme is a multiplier that is fed with the signal itself and the reference signal. The result of this operation is that the signal is shifted to DC, while the

baseline is shifted around 100Hz. Indeed, the original frequency content of the baseline, that varies on a timescale of tens of seconds, is well below 1Hz.
After the multiplier, a low pass filter (LPF) is used to limit the noise and to filter out the baseline. The lower limitation to the frequency of the pole of the LPF is given by the variation of the signal. For an estimation of the signal bandwidth, we can assume an exponential envelope with an overall duration of 1s, corresponding to a time constant of 200ms and thus a bandwidth of about $\frac{1}{2\pi 200ms} \cong 0.8Hz$.
A suitable value for the pole of the LPF is $f_{LPF} = 10Hz$, that is one decade above the estimated signal bandwidth and about one decade below the shifted baseline.
The resulting signal to noise ratio of this acquisition scheme is:

$$\frac{S}{N} = \frac{V_P}{\sqrt{2S_V \frac{\pi}{2} f_{LPF}}} = 56nV$$

C) *Taking a generic gated integrator as an example, describe in detail (from a theoretical point of view) in the time domain the difference between the signal-to-noise ratio obtainable with analogue and digital filtering as the sampling frequency varies.*

The comprehensive theoretical demonstration required in this point is out of the scope of this exercise book. Just to guide the reader, it can be rapidly observed that this question is not specifically related to the scenario described in this problem, but it is rather a general question asking for a comparison between analog and digital filtering with the gated integrator taken just as a reference shape of the weighting function. The intuitive concept of digital filtering approaching the theoretically superior analog filtering for a sampling frequency that tends to infinite should be mathematically demonstrated.

This exercise book collects exam texts of the course Signal Recovery currently offered at Politecnico di Milano within the Master's Degree program in Electronics Engineering. The book is mainly directed to the students of this course. It is intended to provide students with a tool that can help them to bridge the gap between the study of the theory and the solution of real problems. The book provides a detailed discussion and explanation of the solution to the selected problems. Conceived to cover as many aspects of the course as possible, this book is not exhaustive and it cannot be considered in any way sufficient material to learn signal recovery topics.