

## COURSE OUTLINE

- Introduction
- Signals and Noise
- **Filtering: Band-Pass Filters 2 – BPF2**
- Sensors and associated electronics

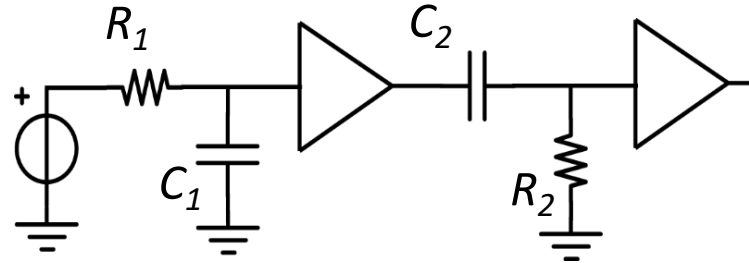
- Band-pass filtering with High-pass plus Low-Pass filters (CR - RC)
- LCR parallel Resonant Filter
- Pro's and Con's of real tuned filters

# Band-pass filtering with High-pass plus Low-Pass filters (CR - RC )

# RC lowpass plus CR highpass = bandpass

## Cascaded two-cell filter:

$$\begin{aligned} \text{low-pass } T_1 &= R_1 C_1 & f_{p1} &= 1/2\pi T_1 \\ \text{high-pass } T_2 &= R_2 C_2 & f_{p2} &= 1/2\pi T_2 \end{aligned}$$



$$H = \frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{jf}{f_{p1}}} \cdot \frac{\frac{jf}{f_{p2}}}{1 + \frac{jf}{f_{p2}}}$$

$$|H|^2 = \frac{1}{1 + \left(\frac{f}{f_{p1}}\right)^2} \cdot \frac{\left(\frac{f}{f_{p2}}\right)^2}{1 + \left(\frac{f}{f_{p2}}\right)^2}$$

With equal poles  $T_1 = T_2 = T$  and  $f_{p1} = f_{p2} = f_p$

$$H = \frac{\frac{jf}{f_p}}{\left(1 + \frac{jf}{f_p}\right)^2}$$

$$|H| = \frac{\frac{f}{f_p}}{1 + \left(\frac{f}{f_p}\right)^2}$$

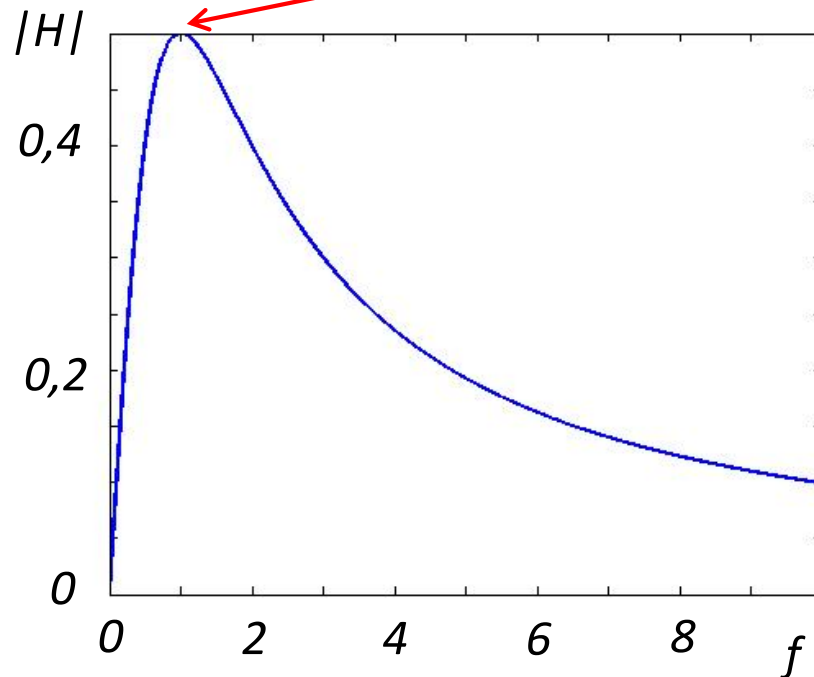
at band center  $f=f_p$  peak value  
and phase zero

$$\begin{cases} |H(f_p)| = \frac{1}{2} \\ \arg H(f_p) = 0 \end{cases}$$

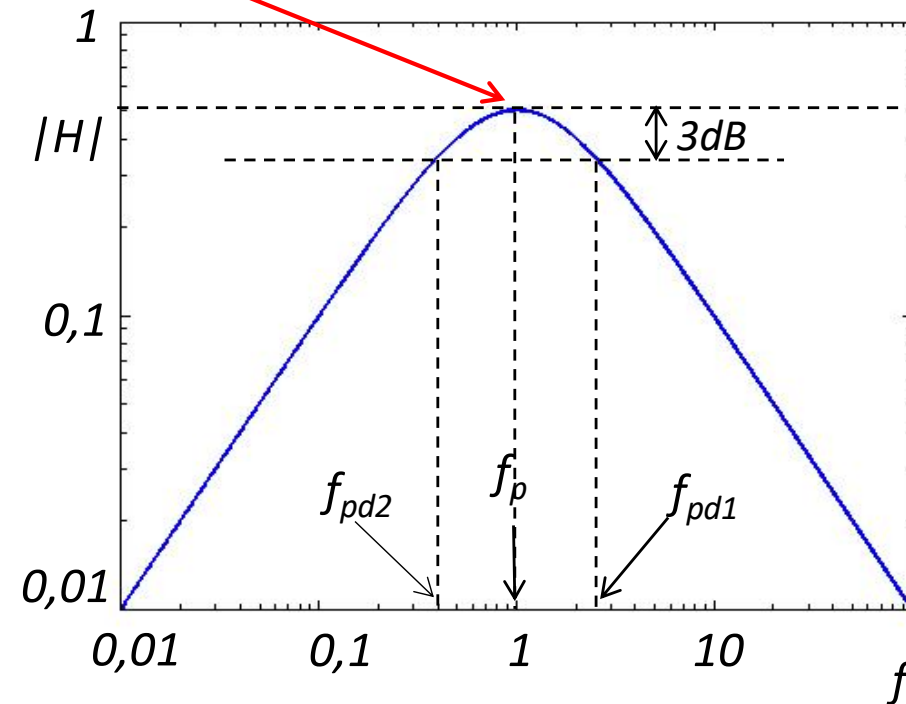
# RC lowpass plus CR highpass = bandpass

Plots of  $|H(f)|$  with  $f_p = 1$

NB: peak is 0,5



Linear – linear plot  
 $|H(f)|$  vs.  $f$



Log – Log plot (Bode plot)  
 $|H(f)|$  vs.  $f$

$$|H| = \frac{\frac{f}{f_p}}{1 + \left(\frac{f}{f_p}\right)^2} \quad \text{peak value at } f=f_p \quad |H(f_p)| = \frac{1}{2}$$

$$\text{3dB down points } f_{pd1} \text{ and } f_{pd2} \quad |H(f_{pd1})| = |H(f_{pd2})| = \frac{|H(f_p)|}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$x = \frac{f}{f_p} \quad \longrightarrow \quad \frac{x}{1+x^2} = \frac{1}{2\sqrt{2}} \quad \longrightarrow \quad x_{1,2} = \sqrt{2} \pm 1$$

$$f_{pd1,pd2} = (\sqrt{2} \pm 1)f_p$$

3dB down pass-band

$$\Delta f_p = f_{pd1} - f_{pd2} = 2f_p$$

NOT narrow-band !!

$$\frac{\Delta f_p}{f_p} = 2$$

From the definition of white noise bandwidth  $\Delta f_n$  (**with unilateral  $S_B$** )



$$\overline{n_B^2} = S_B \cdot |H(f_p)|^2 \cdot \Delta f_n = S_B \cdot \frac{1}{4} \cdot \Delta f_n$$

by comparison with the computed\* output power

$$\overline{n_B^2} = S_B \cdot \int_0^\infty |H(f)|^2 df = S_B \cdot f_p \cdot \frac{\pi}{4}$$

we get

$$\Delta f_n = \pi f_p = \frac{1}{2T} = \frac{\pi}{2} \Delta f_p$$

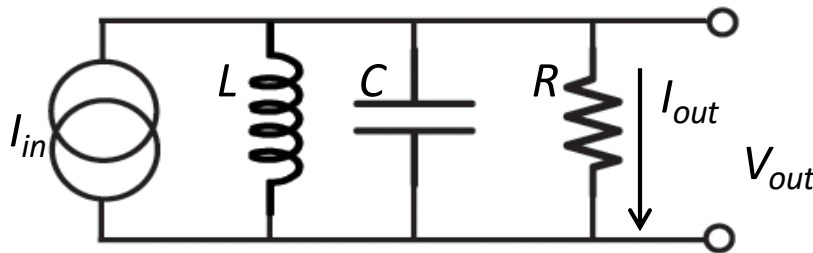
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\* Noise computation

$$\begin{aligned} \overline{n_B^2} &= S_B \cdot \int_0^\infty \frac{\left(\frac{f}{f_p}\right)^2}{\left[1 + \left(\frac{f}{f_p}\right)^2\right]^2} df = S_B f_p \cdot \int_0^\infty \frac{x^2}{[1 + x^2]^2} dx = S_B f_p \cdot \int_0^\infty \frac{2x}{[1 + x^2]^2} \cdot \frac{x}{2} dx \\ &= S_B f_p \cdot \left\{ \left| -\frac{x}{2} \frac{1}{1 + x^2} \right|_0^\infty + \frac{1}{2} \int_0^\infty \frac{1}{1 + x^2} dx \right\} = S_B f_p \cdot \frac{1}{2} |\arctan x|_0^\infty = S_B f_p \cdot \frac{\pi}{4} \end{aligned}$$

# LCR parallel Resonant Filter





$$Z = \frac{V_{out}}{I_{in}} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}} = R \frac{j\omega \frac{L}{R}}{(1 - \omega^2 LC) + j\omega \frac{L}{R}}$$

Denoting by  $H(\omega) = \frac{I_{out}}{I_{in}} = \frac{1}{R} \frac{V_{out}}{I_{in}} = \frac{j\omega \frac{1}{RC}}{\left(\frac{1}{LC} - \omega^2\right) + j\omega \frac{1}{RC}}$  we have  $Z = R \cdot H(\omega)$

At the **resonance frequency**

$$\omega_o = \frac{1}{\sqrt{LC}}$$

the reactive impedances cancel each other

so that the impedance is purely resistive  $Z(\omega_o) = R$

that is  $H(\omega_o) = 1$  and  $\arg H(\omega_o) = 0$

Another basic parameter is the **characteristic resistance**  $R_o$ , which for the oscillation at  $\omega = \omega_o$  represents the ratio

*(amplitude of voltage on C) / (amplitude of current in L)*

$$R_o = \sqrt{\frac{L}{C}}$$

Starting from the poles:

$$s_{1,2} = -\alpha_o \pm \sqrt{\alpha_o^2 - \omega_o^2}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$\alpha_o = \frac{1}{2RC}$$

We can study the behavior of its  $\delta$ -response:

The  $\delta$ -response  $h(t)$  is:

- damped (real poles) if  $\omega_o^2 < \alpha_o^2$ , that is  $R^2 < \left(\frac{R_o}{2}\right)^2$
- critically damped (coincident real poles) if  $\omega_o^2 = \alpha_o^2$ , that is  $R^2 = \left(\frac{R_o}{2}\right)^2$
- oscillatory (complex poles) if  $\omega_o^2 > \alpha_o^2$ , that is  $R^2 > \left(\frac{R_o}{2}\right)^2$

The **higher is R** with respect to  $R_o$ , the **lower is the dissipation**  
and the **slower the damping** of the oscillation

# Resonator Quality Factor Q

The energy  $E$  stored in the circuit oscillates from C to L and back while it decays exponentially due to dissipation in R.

The lower is the loss rate, the higher is the resonator quality.

The reciprocal of this loss rate is defined **Quality Factor Q** of the resonator

that is 
$$-\frac{1}{E} \frac{dE}{d\vartheta} = \frac{1}{Q}$$
 we can calculate 
$$Q = \frac{\omega_o}{2\alpha_o} = \frac{R}{R_o}$$

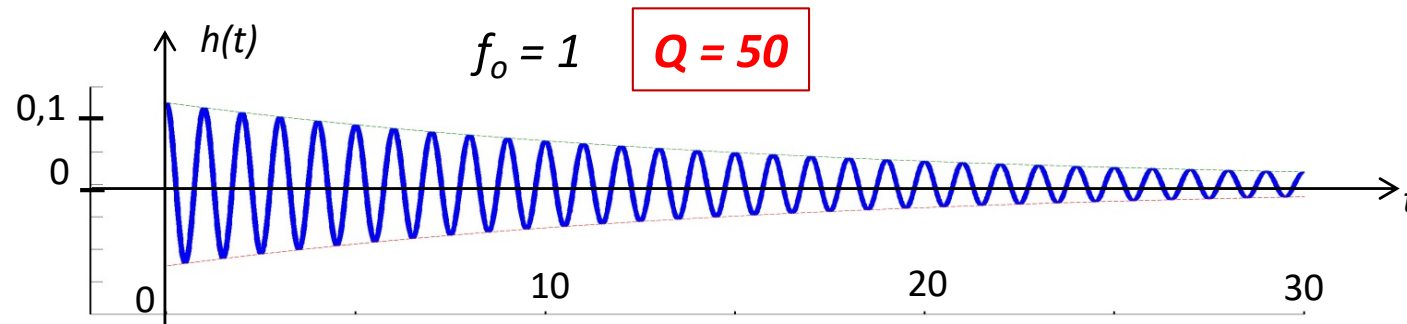
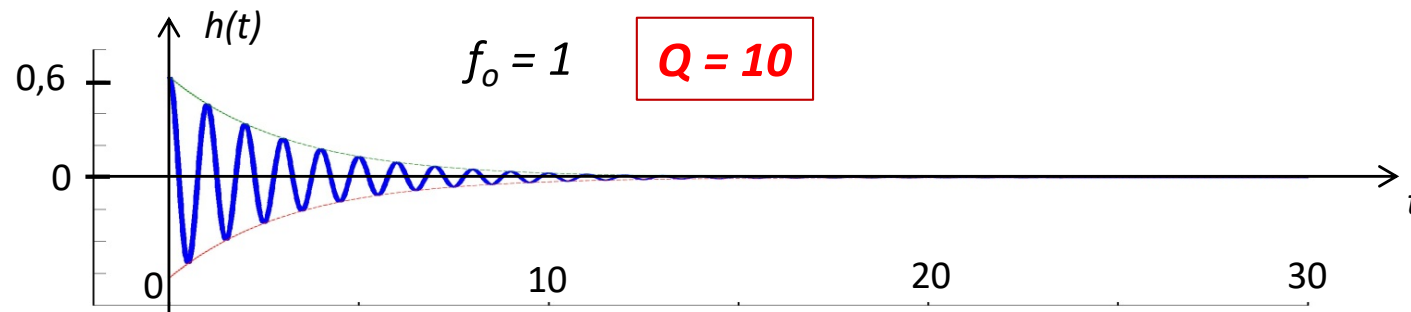
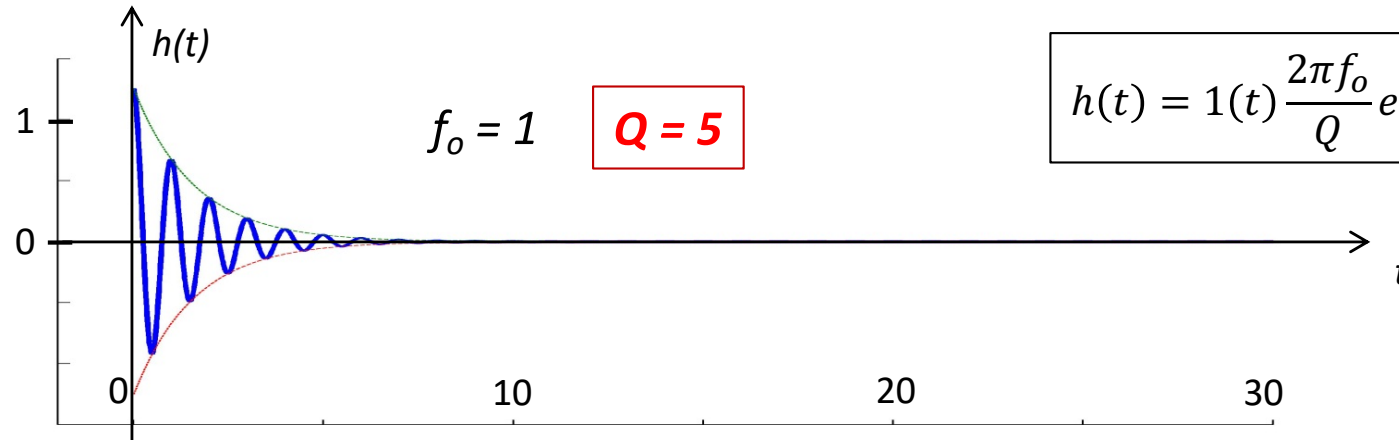
The **higher** is  $R \gg R_o$  the **lower** is the dissipation ( $Q \rightarrow \infty$  for  $R \rightarrow \infty$ )

the transfer can be expressed in terms of resonance frequency  $\omega_o$  and quality factor Q

$$H(\omega) = \frac{j\omega \cdot \frac{\omega_o}{Q}}{(\omega_o^2 - \omega^2) + j\omega \cdot \frac{\omega_o}{Q}}$$

# LRC resonant filter: $\delta$ -response

12



$$\varphi = \arg H(\omega) = \operatorname{arctg} \left[ \frac{Q}{\omega \omega_o} (\omega_o + \omega)(\omega_o - \omega) \right]$$

For  $\omega \rightarrow +\infty$      $|H| \rightarrow 0$      $\varphi = \arg H(\omega_o) \rightarrow -\pi/2$  ( $-90^\circ$ )

For  $\omega \rightarrow -\infty$      $|H| \rightarrow 0$      $\varphi = \arg H(\omega_o) \rightarrow +\pi/2$  ( $+90^\circ$ )

For  $\omega = \omega_o$      $H(\omega_o) = 1$      $\varphi = \arg H(\omega_o) = 0$     and

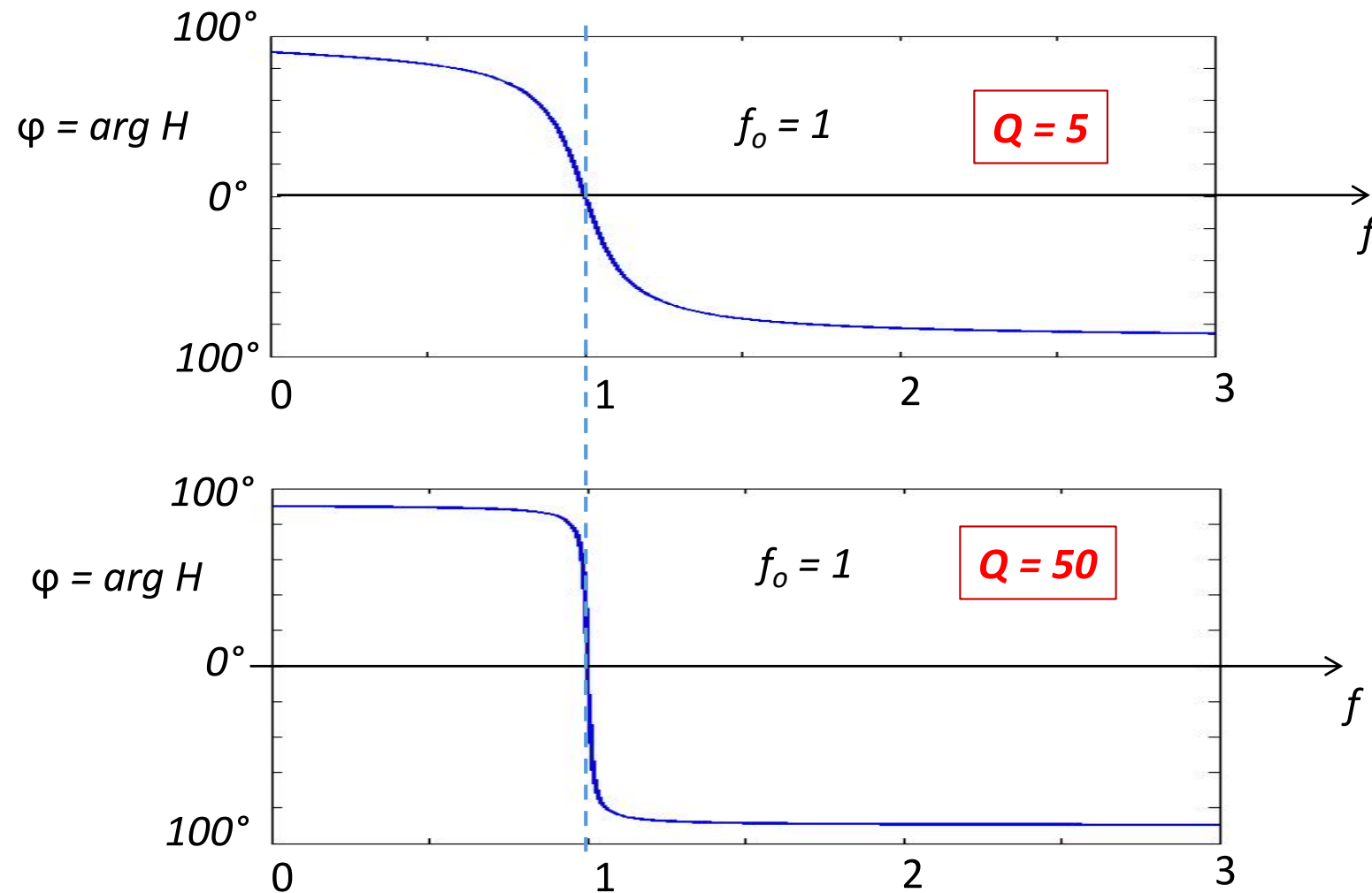
$$\left( \frac{d\varphi}{d\omega} \right)_{\omega=\omega_o} = -2 \frac{Q}{\omega_o}$$

The phase impressed by the filter is exactly zero at exactly the band center, but rapidly increases as  $\omega$  is shifted.

Note that the higher is  $Q$  the steeper is the increase     $\left( \frac{d\varphi}{d\omega} \right) \propto Q$



# LRC resonant filter transfer function: phase



$$|H(\omega)|^2 = \frac{\omega^2 \cdot \frac{\omega_o^2}{Q^2}}{(\omega_o^2 - \omega^2)^2 + \omega^2 \cdot \frac{\omega_o^2}{Q^2}} = \frac{1}{1 + Q^2 \left( \frac{\omega - \omega_o}{\omega_o} \right)^2 \left( \frac{\omega + \omega_o}{\omega} \right)^2}$$

- «**Lower wing**» approximation valid for  $\omega \ll \omega_o$

$$|H(\omega)| \approx |H_L(\omega)| = \frac{\omega}{\omega_o} \frac{1}{Q} \quad \text{i.e.} \quad \boxed{|H_L(\omega)| \propto \omega}$$

- «**Higher wing**» approximation valid for  $\omega \gg \omega_o$

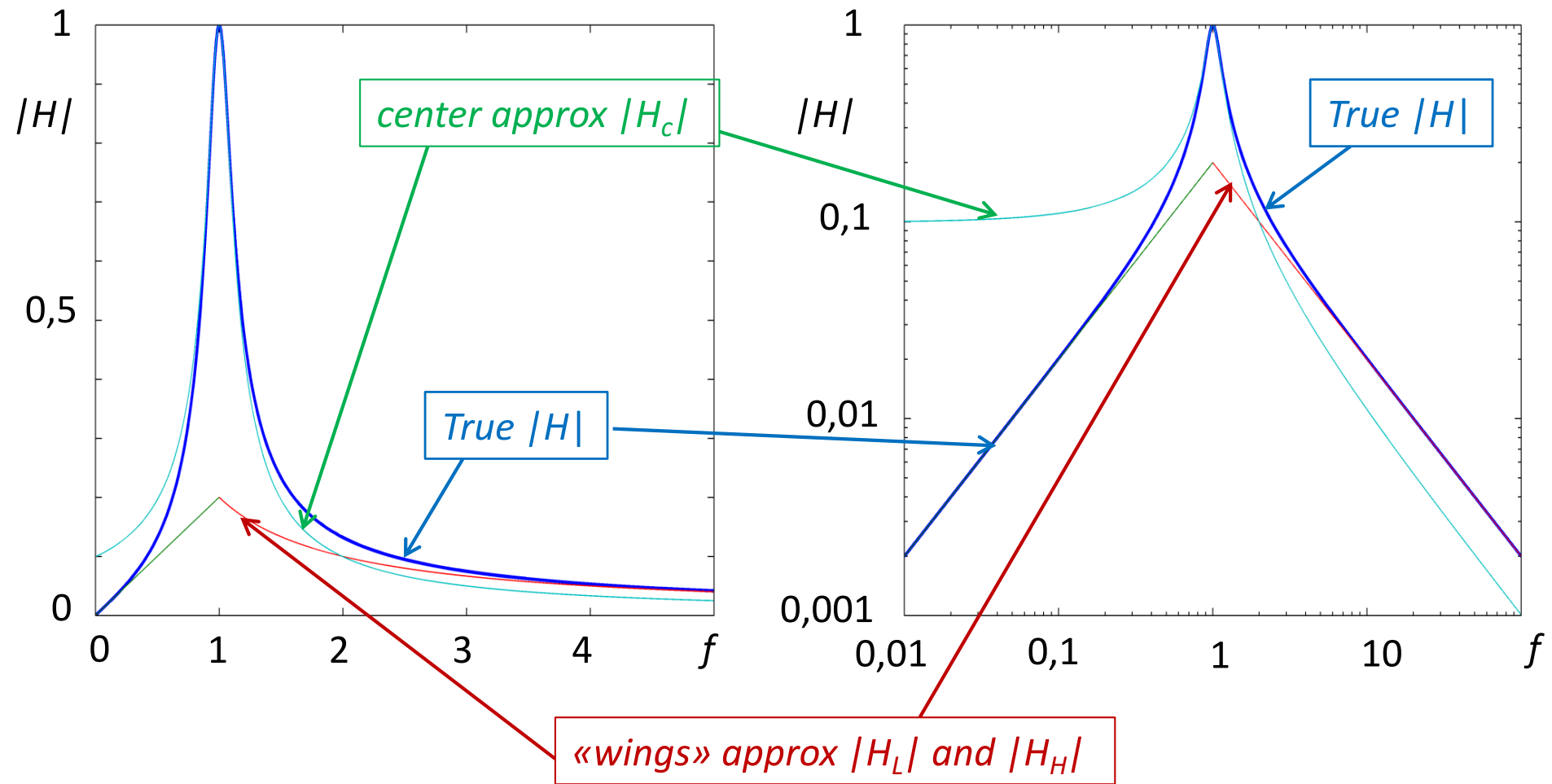
$$|H(\omega)| \approx |H_H(\omega)| = \frac{\omega_o}{\omega} \frac{1}{Q} \quad \text{i.e.} \quad \boxed{|H_H(\omega)| \propto \frac{1}{\omega}}$$

- «**Central lobe**» approximation valid for  $|\omega - \omega_o| \ll \omega_o$ , that is for  $\omega \approx \omega_o$

$$\boxed{|H(\omega)|^2 \approx |H_C(\omega)|^2 = \frac{1}{1 + 4Q^2 \left( \frac{\omega - \omega_o}{\omega_o} \right)^2}}$$

$$f_o = 1$$

$$Q = 5$$



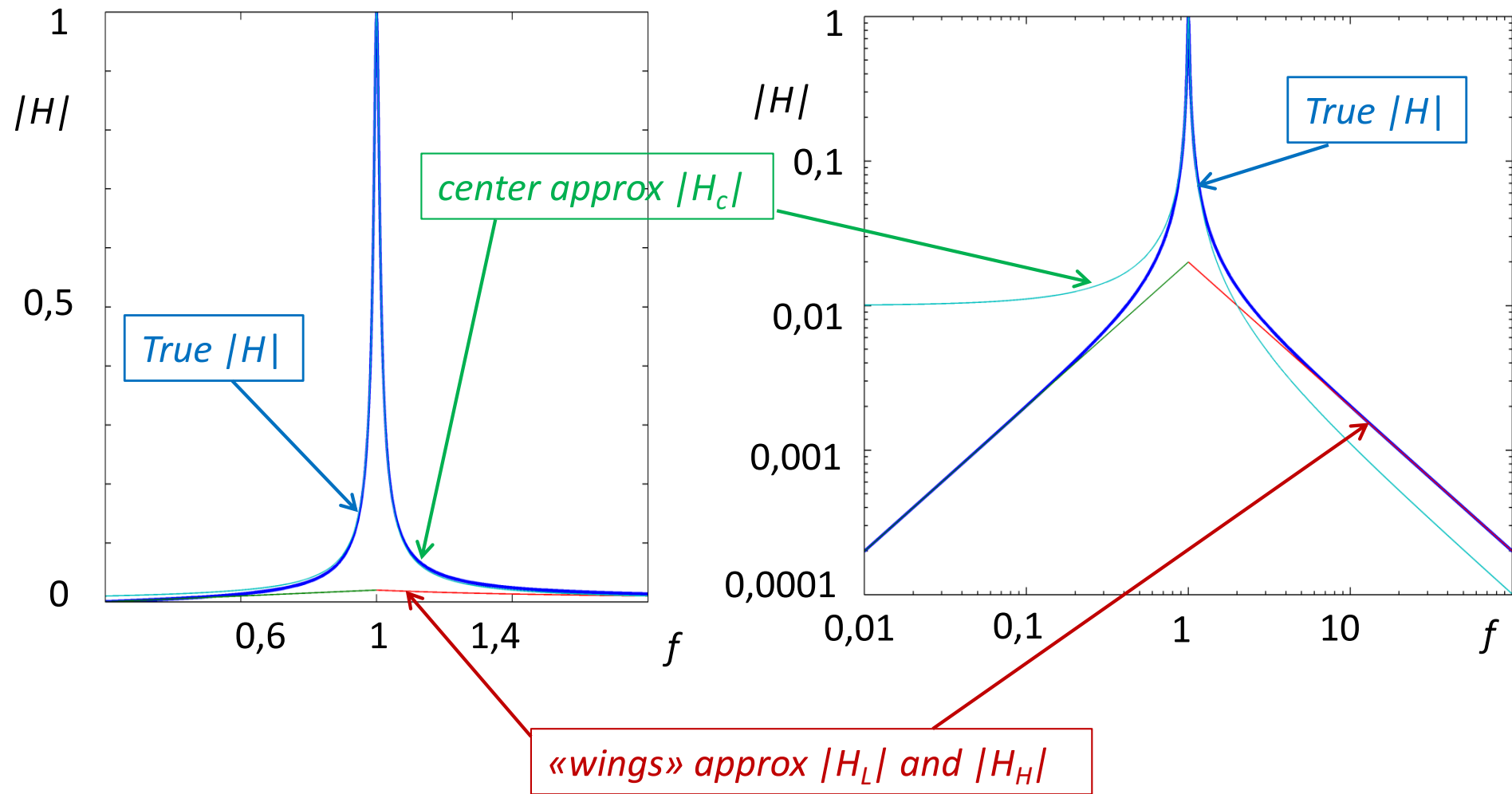


# LRC resonant filter transfer function: module

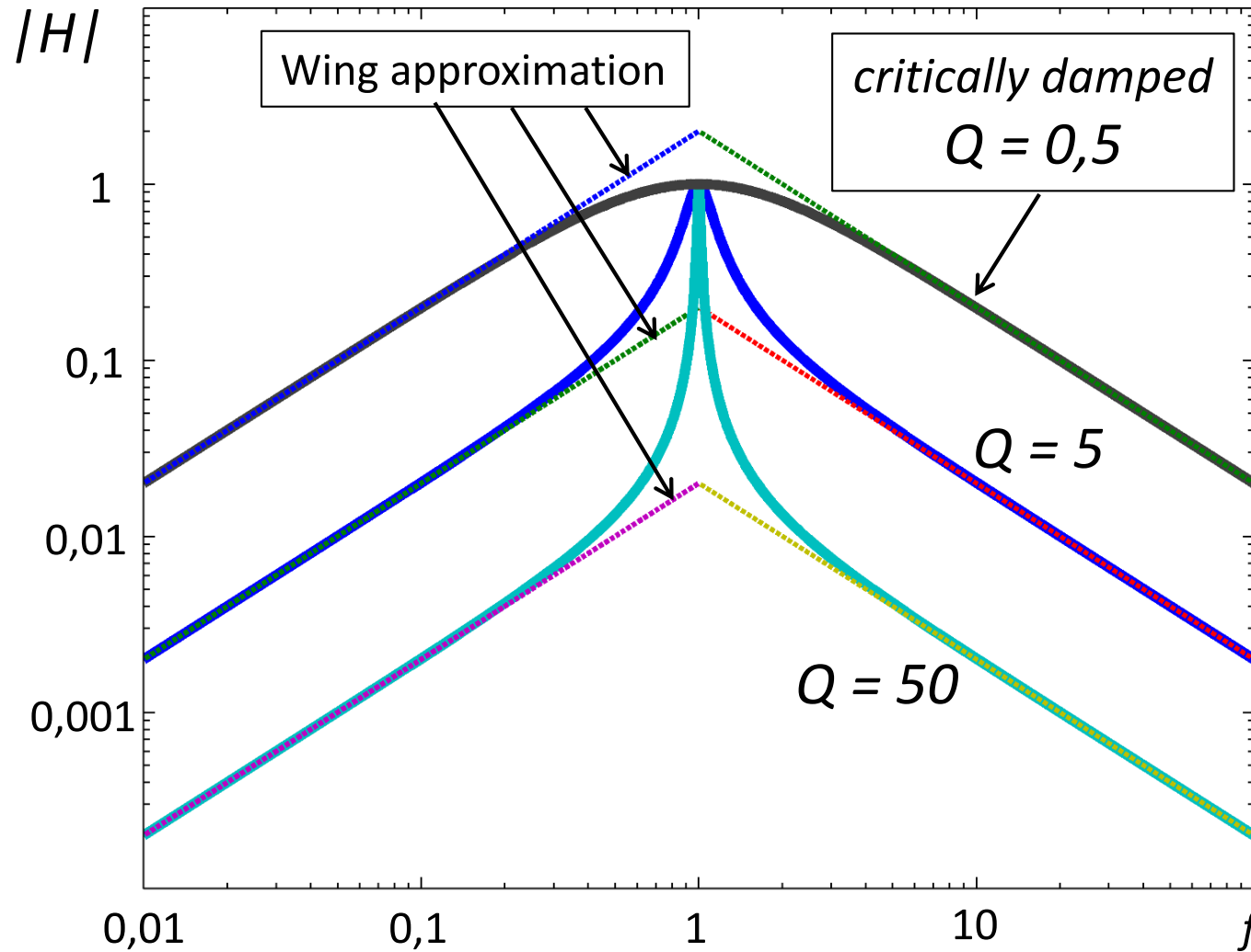
17

$$f_o = 1$$

$$Q = 50$$



## Comparison of filters with different $Q$



**Bandwidth for signals:** defined by the 3dB down points  $\omega_{dL}$  and  $\omega_{dH}$   
where  $|H(\omega_{dL})|^2 = |H(\omega_{dH})|^2 = \frac{1}{2}$

$$\Delta\omega_s = \omega_{dH} - \omega_{dL}$$

For cases with  $Q \gg 1$  we can use the central lobe approximation

$$|H_c(\omega_d)|^2 = \frac{1}{1 + 4Q^2 \left( \frac{\omega_d - \omega_o}{\omega_o} \right)^2} = \frac{1}{2}$$

and we find  $\omega_{dH} - \omega_o = \omega_o - \omega_{dL} = \frac{\omega_o}{2Q}$

The signal bandwidth thus is  $\Delta\omega_s = \frac{\omega_o}{Q}$   $\frac{\Delta\omega_s}{\omega_o} = \frac{\Delta f_s}{f_o} = \frac{1}{Q}$

Two basic advantages with respect to the CR-RC bandpass filter are quite evident:

- **No** signal attenuation at the center frequency
- **Narrow** filtering bandwidth even with moderately high  $Q$  values

The bandwidth for white noise is defined by

$$\Delta f_n = \int_0^\infty |H(f)|^2 df$$

In cases with  $Q \gg 1$  we can use for  $H(f)$  the central lobe approximation and take into account that  $|H_c(f)|^2$  is with good approximation symmetrical with respect to the band center  $f_o$ , thus obtaining

$$\int_0^\infty |H(f)|^2 df \approx 2 \int_{f_o}^\infty |H_c(f)|^2$$

and therefore

$$\Delta f_n = 2 \int_{f_o}^\infty \frac{1}{1 + 4Q^2 \left( \frac{f - f_o}{f_o} \right)^2} df = \frac{f_o}{Q} \int_0^\infty \frac{1}{1 + x^2} dx = \frac{\pi f_o}{2 Q}$$

$$\Delta f_n = \frac{\pi f_o}{2 Q} = \frac{\pi}{2} \Delta f_s$$

# Pro's and Con's of real tuned filters

- **Real capacitors and inductors are not pure C and L.** Their equivalent circuits include also finite resistances that model the internal sources of energy dissipation that inherently limit the Q of resonant circuits.
- In general, the dissipation is higher in components with higher value of L or C. **Good quality capacitors with low dissipation are available from pF to about 1  $\mu$ F.** Inductors are more problematic than capacitors. **Good quality components are available from nH to a few 100nH.** Even components with fairly small L (typically a few 10 nH) have non negligible internal resistance.
- **Stray reactances must not be overlooked.** In discrete circuitry stray capacitances are in the order of pF and stray inductances are in the order of nH. In integrated circuits the values are much smaller, thanks to the very small physical size of the components.
- Since the resonance is at  $f_o = \frac{1}{\sqrt{2\pi LC}}$ , **for obtaining a low frequency  $f_o$  high values of both L and C are required:** in fact, with C=1  $\mu$ F and L= 100 nH one gets  $f_o$  =1,26 MHz. Therefore, the Q values really obtained in the tuned filters progressively decrease as the desired resonant frequency decreases.

- For high frequencies  $f_o > 100\text{MHz}$  values of  $Q > 10$  are currently obtained, up to almost  $Q \approx 100$  with clever design and high quality components.
- For intermediate frequencies  $1\text{MHz} < f_o < 100\text{MHz}$  values up to  $Q \approx 10$  are obtained with careful design and implementation
- For  $f_o < 1\text{MHz}$  it becomes progressively more difficult to obtain high  $Q$  values as the frequency decreases. Anyway, even with moderate  $Q$  the performance of the tuned filters is remarkable and in many practical cases filters with  $Q \approx 5$  are really satisfactory.
- For a given  $Q$ , note that the noise bandwidth is reduced as the resonant frequency  $f_o$  is reduced:  $\Delta f_n = \frac{\pi}{2} \frac{f_o}{Q}$ .

Constant-parameter tuned filters are a simple and economical solution, widely employed in prefiltering stages and other simple situations, but their use in high-performance filtering is hindered by some intrinsic drawbacks.

- **The accuracy and relative stability of  $f_o$  directly depends on that of the C and L values. Drift of  $f_o$  due to aging and temperature must be kept smaller** than the filter bandwidth, in order to avoid uncontrolled variation of the output signal amplitude and phase. This may be really difficult in case of very narrow bandwidth. In particular, strong phase variations are caused by even small variations of  $f_o$  because of the strong  $d\phi/df$  at band-center of filters with high Q
- **Cascading simple filter stages for improving the cutoff characteristics is not practical** for narrow-band filters, because they should have very accurately equal and stable  $f_o$ .
- **The value of C influences both the center frequency  $f_o$  and the bandwidth  $\Delta f_s$ ,** so that it is not easy to design a filter with specified  $f_o$  and specified  $\Delta f_s$ .
- **It is even more difficult to design a filter with adjustable  $f_o$**  and constant bandwidth  $\Delta f_s$ , as it is required for measuring power spectra and for other applications.
- In cases where the frequency of a narrow-band signal is not very stable, a filter with very narrow bandwidth can be employed only if its center frequency can be adjusted to track that of the signal. As above outlined, this is not easy to obtain.