COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering: Band-Pass Filters 2 – BPF2
- Sensors and associated electronics

Band-Pass Filters 2

- Band-pass filtering with High-pass plus Low-Pass filters (CR - RC)
- LCR parallel Resonant Filter
- Pro's and Con's of real tuned filters

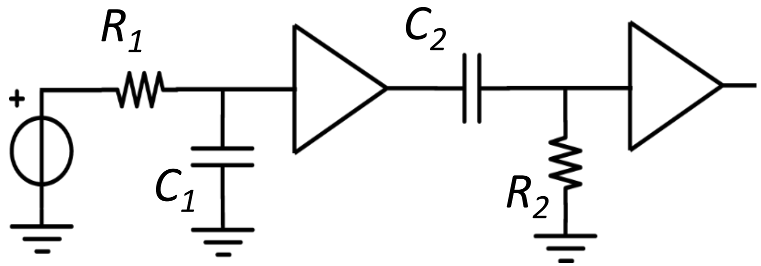
Band-pass filtering with High-pass plus Low-Pass filters (CR - RC)

RC lowpass plus CR highpass = bandpass

Cascaded two-cell filter:

low-pass $T_1 = R_1C_1$ $f_{p1} = 1/2\pi T_1$

high-pass $T_2 = R_2C_2$ $f_{p2} = 1/2\pi T_2$



$$H = \frac{V_{out}}{V_{in}} = \frac{1}{1+\frac{jf}{f_{p1}}} \cdot \frac{\frac{jf}{f_{p2}}}{1+\frac{jf}{f_{p2}}}$$

$$|H|^2 = \frac{1}{1+\left(\frac{f}{f_{p1}}\right)^2} \cdot \frac{\left(\frac{f}{f_{p2}}\right)^2}{1+\left(\frac{f}{f_{p2}}\right)^2}$$

With equal poles $T_1 = T_2 = T$ and $f_{p1} = f_{p1} = f_p$

$$H = \frac{\frac{jf}{f_p}}{\left(1+\frac{jf}{f_p}\right)^2}$$

$$|H| = \frac{\frac{f}{f_p}}{1+\left(\frac{f}{f_p}\right)^2}$$

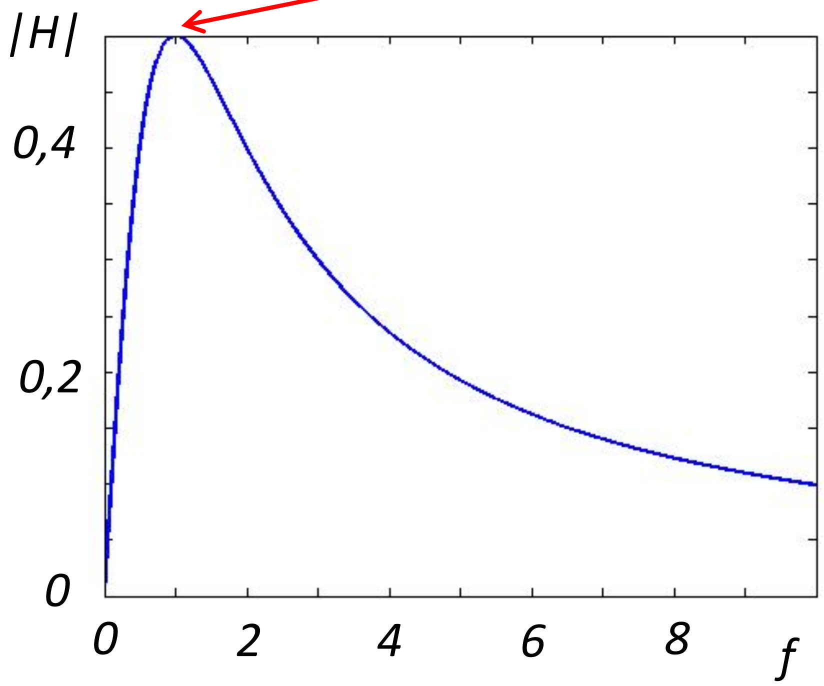
at band center $f=f_p$ peak value and phase zero

$$\begin{cases} |H(f_p)| = \dfrac{1}{2} \\ \arg H(f_p) = 0 \end{cases}$$

RC lowpass plus CR highpass = bandpass

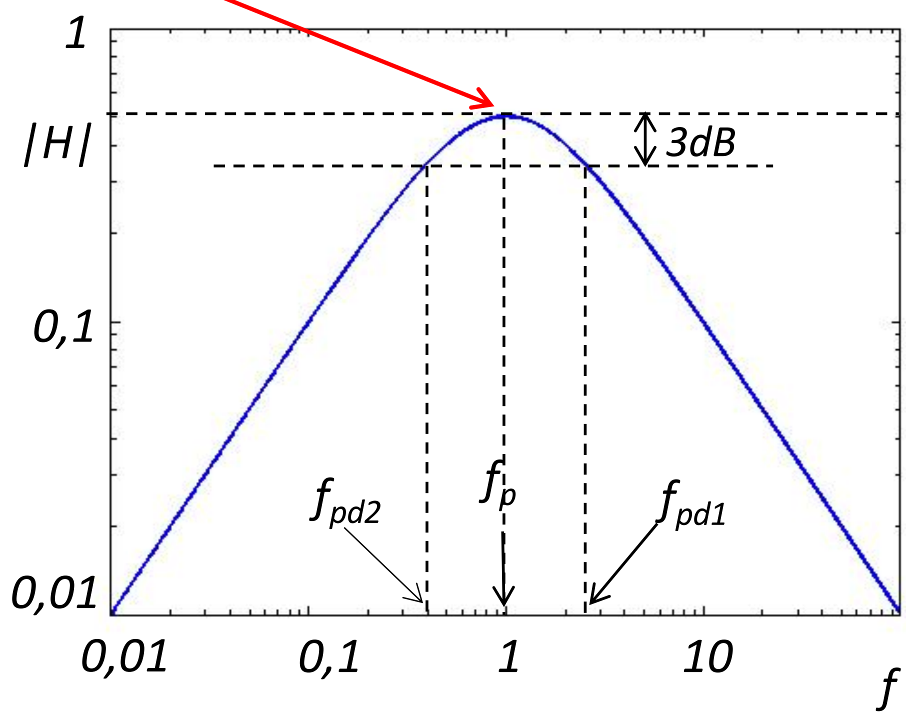
Plots of |H(f)| **with f_p = 1**

NB: peak is 0,5



Linear – linear plot

|H(f)| vs. f



Log – Log plot (Bode plot)

|H(f)| vs. f

3dB-down Bandwidth of CR-RC

$$|H| = \frac{\frac{f}{f_p}}{1+\left(\frac{f}{f_p}\right)^2}$$

peak value at $f=f_p$ $\qquad |H(f_p)| = \frac{1}{2}$

3dB down points f_{pd1} and f_{pd2} $\qquad |H(f_{pd1})| = |H(f_{pd2})| = \frac{|H(f_p)|}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

$$x = \frac{f}{f_p} \longrightarrow \frac{x}{1+x^2} = \frac{1}{2\sqrt{2}} \longrightarrow x_{1,2} = \sqrt{2} \pm 1$$

$$f_{pd1,pd2} = \left(\sqrt{2} \pm 1\right) f_p$$

3dB down pass-band $\qquad \boxed{\Delta f_p = f_{pd1} - f_{pd2} = 2 f_p}$

NOT narrow-band !! $\qquad \boxed{\frac{\Delta f_p}{f_p} = 2}$

White noise bandwidth of CR-RC

From the definition of white noise bandwidth Δf_n (**with unilateral S_B**)

$$\overline{n_B^2} = S_B \cdot \left|H\left(f_p\right)\right|^2 \cdot \Delta f_n = S_B \cdot \frac{1}{4} \cdot \Delta f_n$$

by comparison with the computed* output power

$$\overline{n_B^2} = S_B \cdot \int_0^\infty |H(f)|^2 \, df_n = S_B \cdot f_p \cdot \frac{\pi}{4}$$

we get

$$\boxed{\Delta f_n = \pi f_p = \frac{1}{2T} = \frac{\pi}{2} \Delta f_p}$$

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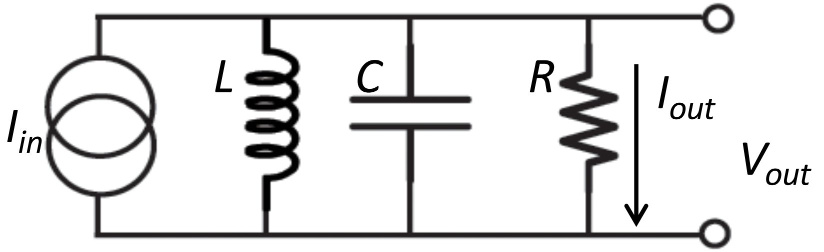
* Noise computation

$$\overline{n_B^2} = S_B \cdot \int_0^\infty \frac{\left(\frac{f}{f_p}\right)^2}{\left[1+\left(\frac{f}{f_p}\right)^2\right]^2} df = S_B f_p \cdot \int_0^\infty \frac{x^2}{[1+x^2]^2} dx = S_B f_p \cdot \int_0^\infty \frac{2x}{[1+x^2]^2} \cdot \frac{x}{2} dx$$

$$= S_B f_p \cdot \left\{ \left| -\frac{x}{2} \frac{1}{1+x^2} \right|_0^\infty + \frac{1}{2} \int_0^\infty \frac{1}{1+x^2} dx \right\} = S_B f_p \cdot \frac{1}{2} |arctg x|_0^\infty = S_B f_p \cdot \frac{\pi}{4}$$

LCR parallel Resonant Filter

LRC parallel resonant filter



$$Z = \frac{V_{out}}{I_{in}} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}} = R\frac{j\omega\frac{L}{R}}{(1-\omega^2 LC) + j\omega\frac{L}{R}}$$

Denoting by $H(\omega) = \frac{I_{out}}{I_{in}} = \frac{1}{R}\frac{V_{out}}{I_{in}} = \frac{j\omega\frac{1}{RC}}{\left(\frac{1}{LC} - \omega^2\right) + j\omega\frac{1}{RC}}$ we have $\boxed{Z = R \cdot H(\omega)}$

At the **resonance frequency** $\boxed{\omega_o = \frac{1}{\sqrt{LC}}}$ the reactive impedances cancel each other

so that the impedance is purely resistive $Z(\omega_o) = R$

that is $H(\omega_o) = 1 \quad and \quad \arg H(\omega_o) = 0$

Another basic parameter is the **characteristic resistance R_o**, which for the oscillation at ω=ω_o represents the ratio

(amplitude of voltage on C) / (amplitude of current in L)

$$\boxed{R_o = \sqrt{\frac{L}{C}}}$$

LRC parallel resonant filter

Starting from the poles: $s_{1,2} = -\alpha_o \pm \sqrt{\alpha_o^2 - \omega_o^2}$ $\omega_o = \dfrac{1}{\sqrt{LC}}$ $\alpha_o = \dfrac{1}{2RC}$

We can study the behavior of its δ-response:

The δ-response h(t) is:

- damped (real poles) if $\omega_o^2 < \alpha_o^2$, that is $R^2 < \left(\dfrac{R_o}{2}\right)^2$
- critically damped (coincident real poles) if $\omega_o^2 = \alpha_o^2$, that is $R^2 = \left(\dfrac{R_o}{2}\right)^2$
- oscillatory (complex poles) if $\omega_o^2 > \alpha_o^2$, that is $R^2 > \left(\dfrac{R_o}{2}\right)^2$

The **higher is R** with respect to R_o, the **lower is the dissipation**

and the **slower the damping** of the oscillation

Resonator Quality Factor Q

The energy *E* stored in the circuit oscillates from C to L and back while it decays exponentially due to dissipation in R.

The lower is the loss rate, the higher is the resonator quality.
The reciprocal of this loss rate is defined **Quality Factor Q** of the resonator

that is $\boxed{-\frac{1}{E}\frac{dE}{d\vartheta} = \frac{1}{Q}}$ we can calculate $\boxed{Q = \frac{\omega_o}{2\alpha_o} = \frac{R}{R_o}}$

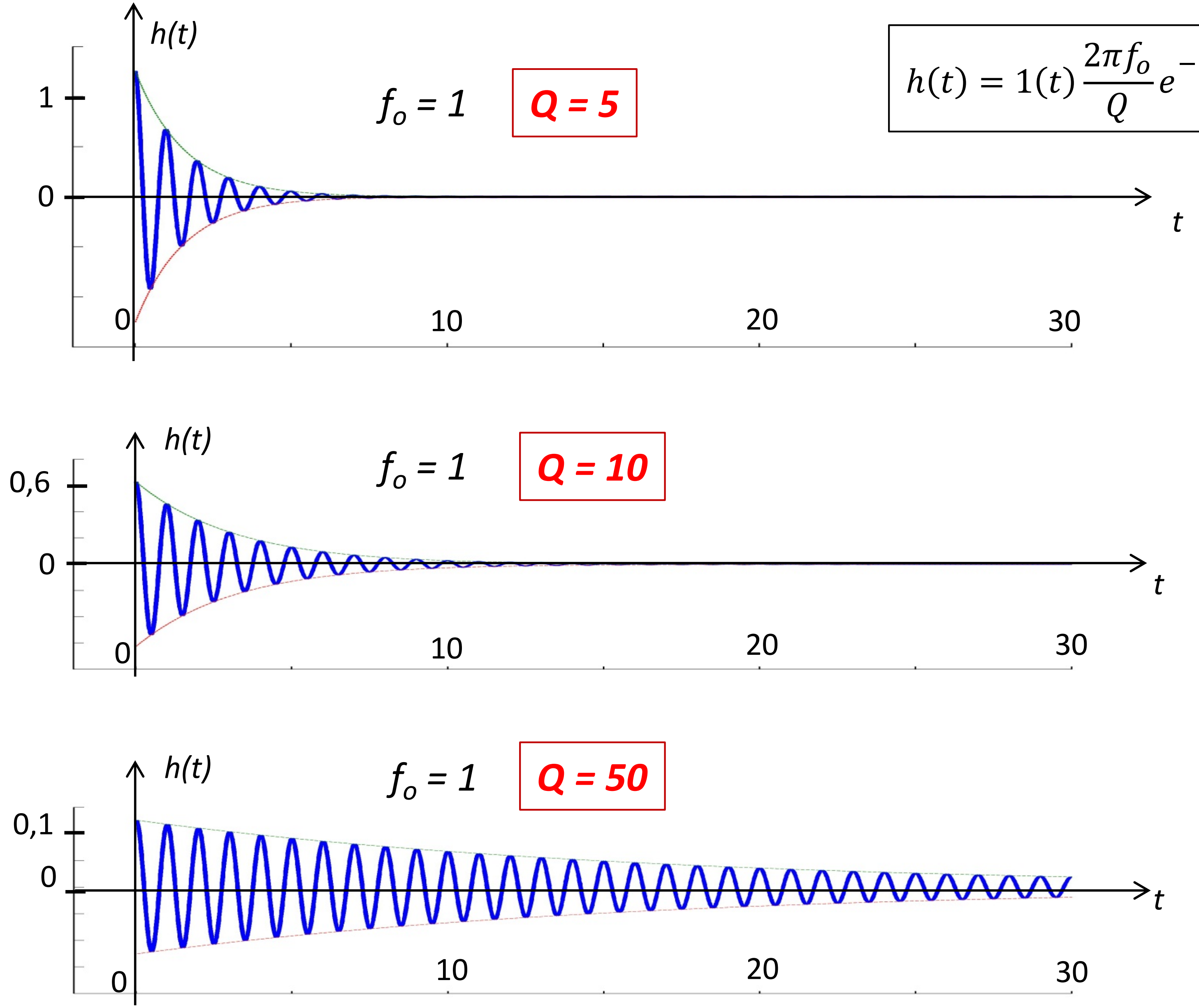
The **higher** is R >> R_o the lower is the dissipation (Q → ∞ for R → ∞)

the transfer can be expressed in terms of resonance frequency ω_o and quality factor Q

$$\boxed{H(\omega) = \frac{j\omega \cdot \frac{\omega_o}{Q}}{(\omega_o^2 - \omega^2) + j\omega \cdot \frac{\omega_o}{Q}}}$$

LRC resonant filter: δ-response

$$h(t) = 1(t)\,\frac{2\pi f_o}{Q}\,e^{-\frac{\pi f_o}{Q}t}\cos\omega_o t$$



LRC resonant filter transfer function: phase

$$\varphi = \arg H(\omega) = arctg\left[\frac{Q}{\omega\omega_o}(\omega_o+\omega)(\omega_o-\omega)\right]$$

For $\omega \rightarrow +\infty$ $|H| \rightarrow 0$ $\varphi = \arg H(\omega_o) \rightarrow -\pi/2$ (- 90°)

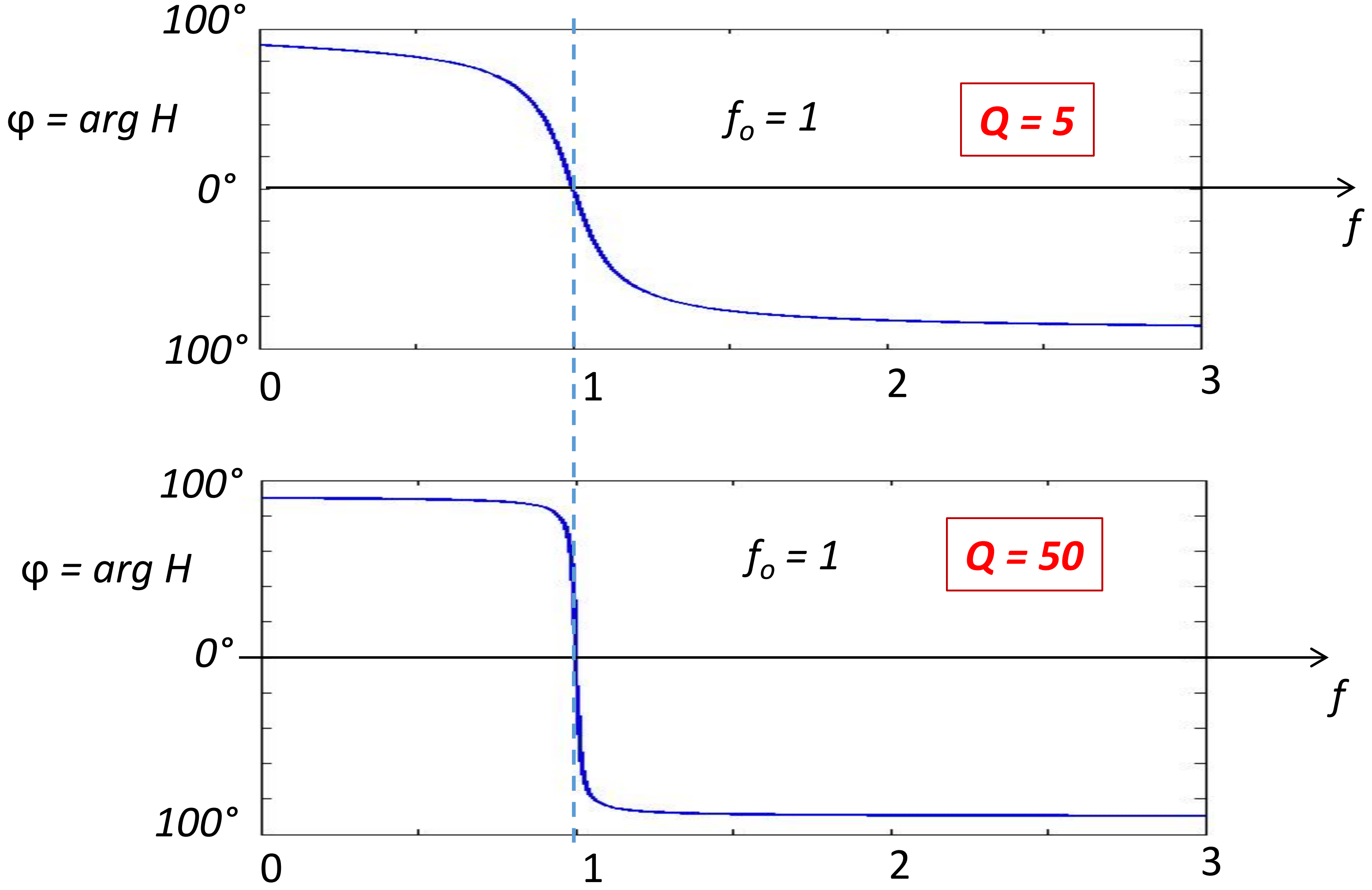
For $\omega \rightarrow -\infty$ $|H| \rightarrow 0$ $\varphi = \arg H(\omega_o) \rightarrow +\pi/2$ (90°)

For $\omega = \omega_o$ $H(\omega_o) = 1$ $\varphi = \arg H(\omega_o) = 0$ and $$\left(\frac{d\varphi}{d\omega}\right)_{\omega=\omega_o} = -2\frac{Q}{\omega_o}$$

The phase impressed by the filter is exactly zero at exactly the band center, but rapidly increases as ω is shifted.

Note that the higher is Q the steeper is the increase $\left(\frac{d\varphi}{d\omega}\right) \propto Q$

LRC resonant filter transfer function: phase



LRC resonant filter transfer function: module

$$|H(\omega)|^2 = \frac{\omega^2 \cdot \frac{\omega_o^2}{Q^2}}{(\omega_o^2 - \omega^2)^2 + \omega^2 \cdot \frac{\omega_o^2}{Q^2}} = \frac{1}{1 + Q^2 \left(\frac{\omega - \omega_o}{\omega_o}\right)^2 \left(\frac{\omega + \omega_o}{\omega}\right)^2}$$

- **«Lower wing»** approximation valid for $\omega << \omega_o$

$$|H(\omega)| \approx |H_L(\omega)| = \frac{\omega}{\omega_o}\frac{1}{Q} \qquad \text{i.e.} \qquad \boxed{|H_L(\omega)| \propto \omega}$$

- **«Higher wing»** approximation valid for $\omega >> \omega_o$

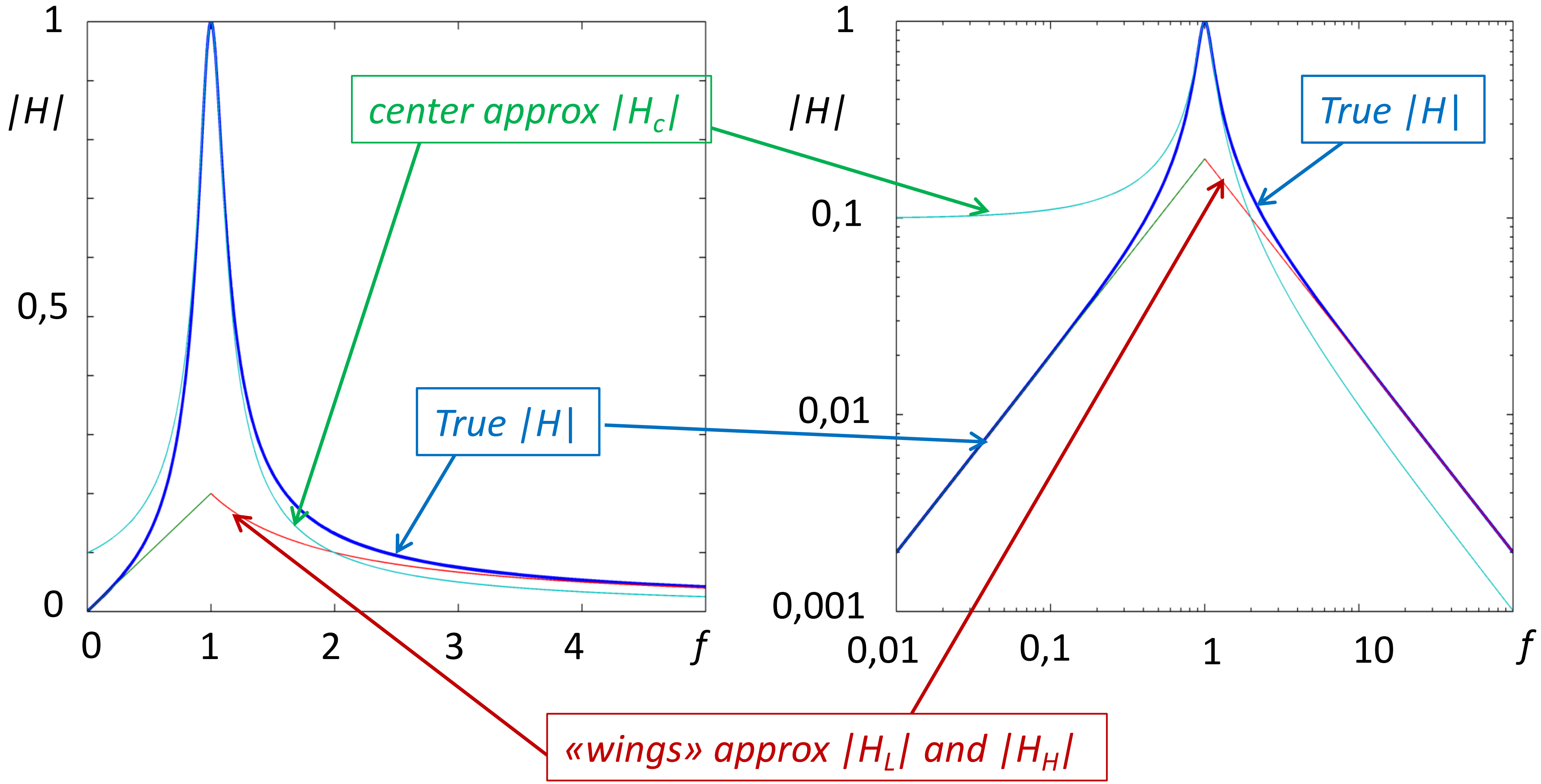
$$|H(\omega)| \approx |H_H(\omega)| = \frac{\omega_o}{\omega}\frac{1}{Q} \qquad \text{i.e.} \qquad \boxed{|H_H(\omega)| \propto \frac{1}{\omega}}$$

- **«Central lobe»** approximation valid for $|\omega - \omega_o| << \omega_o$, that is for $\omega \approx \omega_o$

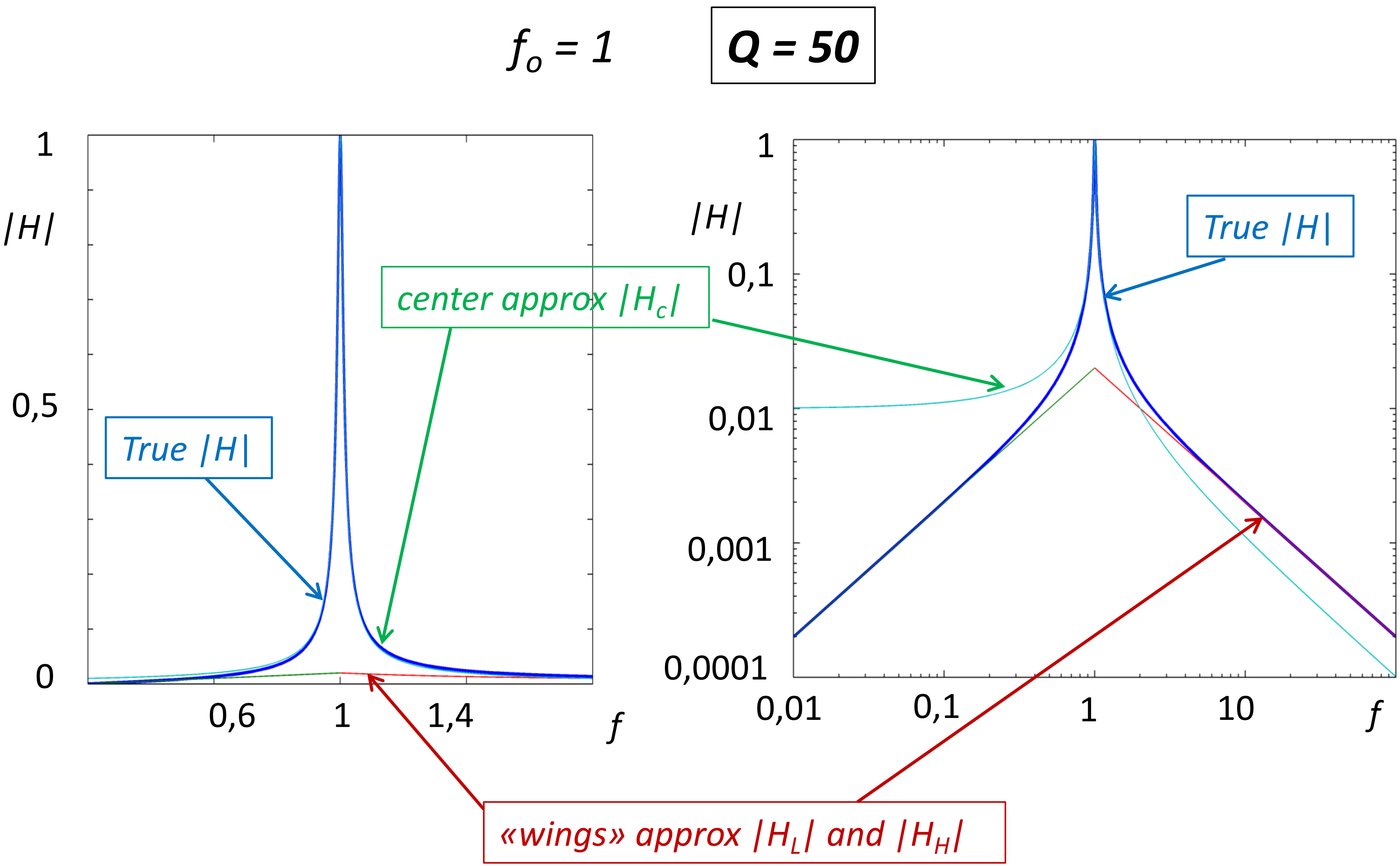
$$\boxed{|H(\omega)|^2 \approx |H_C(\omega)|^2 = \frac{1}{1 + 4Q^2 \left(\frac{\omega - \omega_o}{\omega_o}\right)^2}}$$

LRC resonant filter transfer function: module

$f_o = 1$ $\boxed{Q = 5}$

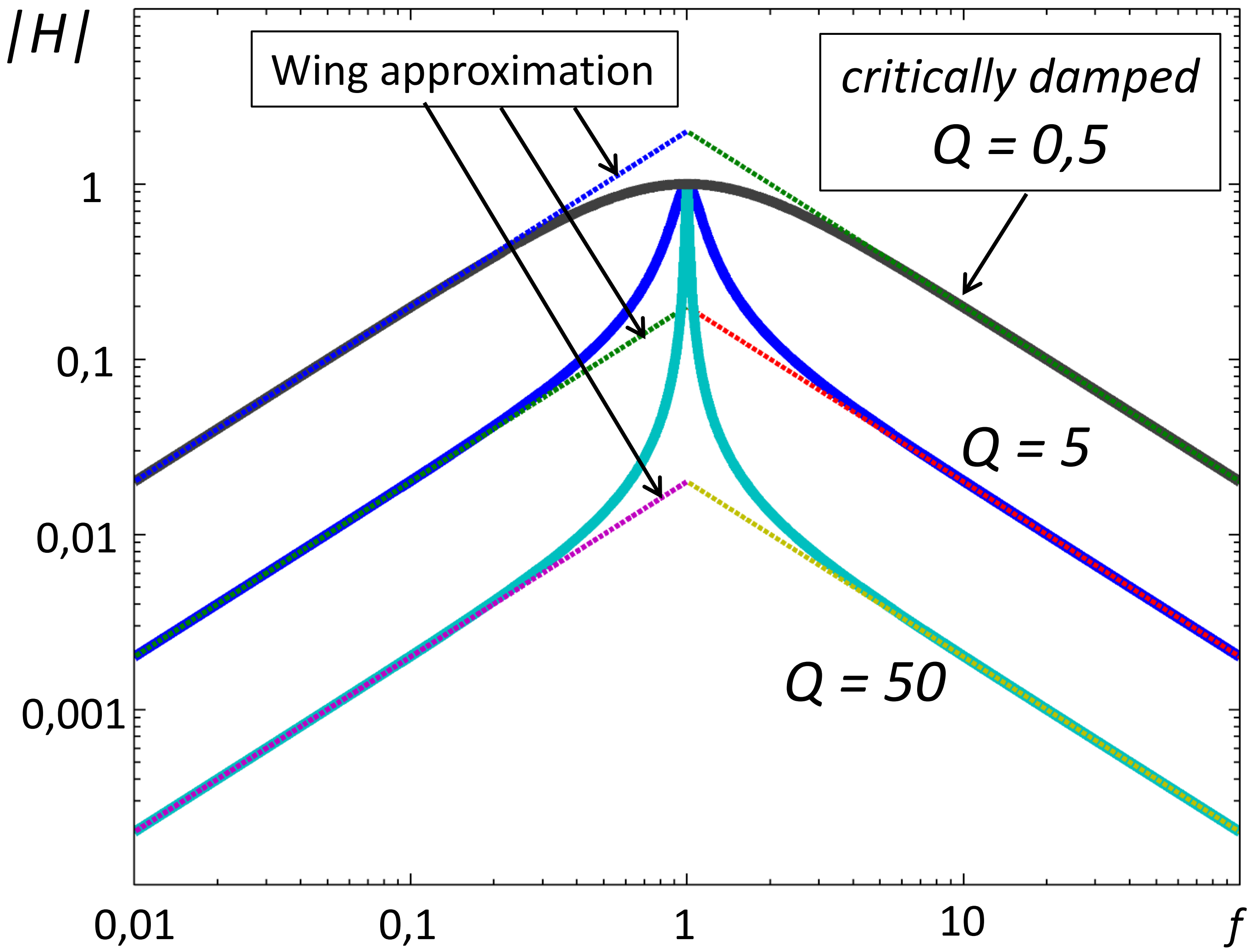


LRC resonant filter transfer function: module



LRC resonant filter transfer function module

Comparison of filters with different Q



LRC resonant filter: signal bandpass

Bandwidth for signals: defined by the 3dB down points ω_{dL} and ω_{dH} where $|H(\omega_{dL})|^2 = |H(\omega_{dH})|^2 = ½$

$$\Delta\omega_s = \omega_{dH} - \omega_{dL}$$

For cases with Q >> 1 we can use the central lobe approximation

$$|H_C(\omega_d)|^2 = \frac{1}{1 + 4Q^2\left(\frac{\omega_d - \omega_o}{\omega_o}\right)^2} = \frac{1}{2}$$

and we find $$\omega_{dH} - \omega_o = \omega_o - \omega_{dL} = \frac{\omega_o}{2Q}$$

The signal bandwidth thus is $$\boxed{\Delta\omega_s = \frac{\omega_o}{Q}}$$ $$\boxed{\frac{\Delta\omega_s}{\omega_o} = \frac{\Delta f_s}{f_o} = \frac{1}{Q}}$$

Two basic advantages with respect to the CR-RC bandpass filter are quite evident:

- **No** signal attenuation at the center frequency
- **Narrow** filtering bandwidth even with moderately high Q values

LRC resonant filter: noise bandpass

The bandwidth for white noise is defined by

$$\Delta f_n = \int_0^{\infty} |H(f)|^2 \, df$$

In cases with Q >> 1 we can use for H(f) the central lobe approximation and take into account that $|H_c(f)|^2$ is with good approximation symmetrical with respect to the band center f_0, thus obtaining

$$\int_0^{\infty} |H(f)|^2 \, df \approx 2 \int_{f_o}^{\infty} |H_C(f)|^2$$

and therefore

$$\Delta f_n = 2 \int_{f_o}^{\infty} \frac{1}{1 + 4Q^2 \left(\frac{f - f_o}{f_o}\right)^2} \, df = \frac{f_o}{Q} \int_0^{\infty} \frac{1}{1 + x^2} \, dx = \frac{\pi}{2} \frac{f_o}{Q}$$

$$\boxed{\Delta f_n = \frac{\pi}{2} \frac{f_o}{Q} = \frac{\pi}{2} \Delta f_s}$$

Pro’s and Con’s of real tuned filters

Issues, Pro's and Con's of real tuned filters

- **Real capacitors and inductors are not pure C and L**. Their equivalent circuits include also finite resistances that model the internal sources of energy dissipation that inherently limit the Q of resonant circuits.

- In general, the dissipation is higher in components with higher value of L or C. **Good quality capacitors with low dissipation are available from pF to about 1 µF**. Inductors are more problematic than capacitors. **Good quality components are available from nH to a few 100nH**. Even components with fairly small L (typically a few 10 nH) have non negligible internal resistance.

- **Stray reactances must not be overlooked**. In discrete circuitry stray capacitances are in the order of pF and stray inductances are in the order of nH. In integrated circuits the values are much smaller, thanks to the very small physical size of the components.

- Since the resonance is at $f_o = \frac{1}{\sqrt{2\pi LC}}$, **for obtaining a low frequency f_o high values of both L and C are required**: in fact, with C=1 µF and L= 100 nH one gets f_o =1,26 MHz. Therefore, the Q values really obtained in the tuned filters progressively decrease as the desired resonant frequency decreases.

Issues, Pro's and Con's of real tuned filters

- **For high frequencies f_o > 100MHz values of Q > 10 are currently obtained, up to almost Q ≈ 100** with clever design and high quality components.
- **For intermediate frequencies 1MHz < f_o < 100MHz values up to Q ≈ 10** are obtained with careful design and implementation
- **For f_o < 1MHz it becomes progressively more difficult to obtain high Q values** as the frequency decreases. Anyway, even with moderate Q the performance of the tuned filters is remarkable and in many practical cases filters with **Q ≈ 5** are really satisfactory.
- For a given Q, note that the noise bandwidth is reduced as the resonant frequency f_o is reduced: $\Delta f_n = \frac{\pi}{2}\frac{f_o}{Q}$.

Constant-parameter tuned filters are a simple and economical solution, widely employed in prefiltering stages and other simple situations, but their use in high-performance filtering is hindered by some intrinsic drawbacks.

Issues, Pro's and Con's of real tuned filters

- **The accuracy and relative stability of f_o directly depends on that of the C and L values. Drift of f_o due to aging and temperature must be kept smaller** than the filter bandwidth, in order to avoid uncontrolled variation of the output signal amplitude and phase. This may be really difficult in case of very narrow bandwidth. In particular, strong phase variations are caused by even small variations of f_o because of the strong dφ/df at band-center of filters with high Q
- **Cascading simple filter stages for improving the cutoff characteristics is not practical** for narrow-band filters, because they should have very accurately equal and stable f_o.
- **The value of C influences both the center frequency f_o and the bandwidth Δf_s** , so that it is not easy to design a filter with specified f_o and specified Δf_s .
- **It is even more difficult to design a filter with adjustable f_o** and constant bandwidth Δf_s , as it is required for measuring power spectra and for other applications.
- In cases where the frequency of a narrow-band signal is not very stable, a filter with very narrow bandwidth can be employed only if its center frequency can be adjusted to track that of the signal. As above outlined, this is not easy to obtain.