Sensors, Signals and Noise

COURSE OUTLINE

• Introduction
• Signals and Noise
• Filtering: Band-Pass Filters 1 - BPF1
• Sensors and associated electronics
Band-Pass Filters 1

- Narrow-Band Signals
- Recovering Narrow-Band Signals from Noise
- Moving Signals in Frequency (Signal Modulation)
Narrow-Band Signals
Narrow-Band Signals

Power signals with a narrow power spectrum, that is, a peak with

- center-frequency $f_s$
- bandwidth $\Delta f_s$ which is small in absolute value, typically $\Delta f_s < 10$ Hz, and/or with respect to the center frequency $\Delta f_s << f_s$

They approximate well a sinusoid over a wide time interval $T_s \approx 1/\Delta f_s$

QUESTION: how can we measure such narrow-band signals in presence of intense white noise? And what if also $1/f$ noise is present?
Recovering Narrow-Band Signals from Noise
Recovering Narrow-Band Signals from noise

Let’s see some typical examples of signals with

- narrow linewidth $\Delta f_s = 1 \text{ Hz}$
- small amplitude $V_s \leq 100 \text{ nV}$

for bringing them to higher level (suitable for processing circuits: filters, meters, etc.) they are amplified by a DC-coupled wide-band preamplifier with

- upper band-limit $f_h = 1 \text{ MHz}$
- noise spectral density (referred to input) with
  - «white» component $\sqrt{S_b} = 5 \text{nV} / \sqrt{\text{Hz}}$
  - and $1/f$ component with corner frequency $f_c = 2 \text{kHz}$

Let us consider three cases with different center-frequency $f_s$:

- **Case 1:** high frequency $f_s = 100 \text{ kHz}$
- **Case 2:** moderately low frequency $f_s = 1 \text{ kHz}$
- **Case 3:** low frequency $f_s = 10 \text{ Hz}$
Recovering Narrow-Band Signals

CASE 1: signal $V_s \leq 100$ $nV$ at high frequency $f_s = 100$ kHz

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal to be recovered is at frequency $f_s = 100$ kHz much higher than the noise corner frequency $f_c = 2$kHz, so that we can use a simple high-pass filter with band-limit $f_i = 10$kHz to cut off the 1/f noise and obtain a rms noise (referred to the preamp input)

$$\sqrt{V_n^2} = \sqrt{S_b} \cdot \sqrt{(f_h - f_i)} \approx \sqrt{S_b} \cdot \sqrt{f_h} = 5 \mu V$$

and therefore

$$\frac{S}{N} = \frac{V_s}{\sqrt{V_n^2}} \leq 0.02 << 1$$

Even the highest signal $V_s = 100$ $nV$ is practically invisible on the oscilloscope display! The noise covers a band $\approx 5 \times$ rms value $\approx 20 \mu V$ and the sinusoidal signal is buried in it!

![Oscilloscope waveform](image)

Vertical scale 50$\mu$V/div
Horizontal scale 5$\mu$s/div
CASE 1: signal $V_s \leq 100 \text{ nV}$ at high frequency $f_s = 100 \text{ kHz}$

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

**SIGNAL:** the power $P_s = \frac{V_s^2}{2} = 50 \cdot 10^{-16} \text{ V}^2$ is within a bandwidth $\Delta f_s = 1 \text{ Hz}$

so that the effective power density of the signal is $\sqrt{S_s} \approx \sqrt{\frac{P_s}{\Delta f_s}} = 70 \text{nV} / \sqrt{\text{Hz}}$

**NOISE:** the effective power density at $f_s = 100 \text{ kHz}$ is $\sqrt{S_b} = 5 \text{nV} / \sqrt{\text{Hz}}$

On the spectrum analyzer display the signal peak is very well visible above the noise!

$$\frac{\sqrt{S_s}}{\sqrt{S_b}} = 14 \gg 1$$

**Conclusion:** good S/N can be obtained with a bandpass filter having bandwidth $\Delta f_b$ matched to the signal band $\Delta f_b \approx \Delta f_s$

$$\frac{S}{N} = \sqrt{\frac{P_s}{S_b \Delta f_b}} = \sqrt{\frac{S_s \Delta f_s}{S_b \Delta f_b}} \approx \sqrt{\frac{S_s}{S_b}} = 14 \gg 1$$
Recovering Narrow-Band Signals

CASE 2: signal $V_s \leq 100 \text{ nV}$ at moderately low frequency $f_s = 1 \text{ kHz}$

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at $f_s = 1 \text{ kHz}$ just below the corner frequency $f_c = 2 \text{ kHz}$.

For reducing the 1/f noise we can still use a high-pass filter, but in order to pass the signal
the band-limit $f_i$ must be reduced: $f_i << f_s = 1 \text{ kHz}$, typically $f_i = 100 \text{ Hz}$.

The rms noise referred to the input is

$$
\sqrt{\frac{V^2}{n}} \approx \sqrt{S_b \left(f_h - f_i\right) + S_b f_c \ln \left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h + S_b f_c \ln \left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h} \approx 5 \mu V
$$

and therefore

$$
\frac{S}{N} = \frac{V_s}{\sqrt{\frac{V^2}{n}}} \leq 0.02 << 1
$$

1/f noise is negligible $S_b f_c \ln \left(\frac{f_h}{f_i}\right) \ll S_b f_h$

The situation is practically equal to that of Case 1: the signal is **practically invisible**
on the oscilloscope display, it’s buried in the noise!
Recovering Narrow-Band Signals

**CASE 2:** signal $V_s \leq 100 \text{nV}$ at moderately low frequency $f_s = 1 \text{kHz}$

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

**SIGNAL:** the power $P_S = \frac{V_S^2}{2} = 50 \cdot 10^{-16} \text{V}^2$ is within a bandwidth $\Delta f_S = 1 \text{Hz}$

so that the effective power density of the signal is $\sqrt{S_S} \approx \sqrt{\frac{P_S}{\Delta f_S}} = 70 \text{nV}/\sqrt{\text{Hz}}$

**NOISE:** due to the 1/f noise, the effective power density at $f_s = 1 \text{kHz}$ is somewhat higher

$$\sqrt{S_n(f_s)} = \sqrt{S_b + S_b \frac{f_c}{f_s}} = \sqrt{S_b} \sqrt{1 + \frac{f_c}{f_s}} = \sqrt{3} \cdot \sqrt{S_b} \approx 8.7 \text{nV}/\sqrt{\text{Hz}}$$

Anyway, on the spectrum analyzer display the signal peak is still well visible above the noise

$$\frac{\sqrt{S_S}}{\sqrt{S_n(f_s)}} = 8 > 1$$

**Conclusion:** a bandpass filter with bandwidth $\Delta f_b$ matched to the signal $\Delta f_b \approx \Delta f_S$

still gives a fairly good S/N

$$\frac{S}{N} = \sqrt{\frac{P_S}{S_n(f_s) \Delta f_b}} = \sqrt{\frac{S_S \Delta f_S}{S_n(f_s) \Delta f_b}} \approx \frac{\sqrt{S_S}}{\sqrt{S_n(f_s)}} = 8 > 1$$
CASE 3: signal $V_s \leq 100 \text{ nV}$ at low frequency $f_s = 10 \text{ Hz}$

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at $f_s = 10 \text{ Hz}$ much below the corner frequency $f_c = 2k\text{Hz}$.

For reducing the the $1/f$ noise we can still use a high-pass filter, but with strongly reduced band-limit $f_i \ll f_s = 10 \text{ Hz}$, typically $f_i = 1 \text{ Hz}$. The rms noise referred to input is

$$\sqrt{\frac{V_n^2}{S}} \approx \sqrt{S_b \left(f_h - f_i\right) + S_b f_c \ln \left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h + S_b f_c \ln \left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h} \approx 5 \mu\text{V}$$

1/f noise is negligible $S_b f_c \ln \left(\frac{f_h}{f_i}\right) \ll S_b f_h$

and therefore

$$\frac{S}{N} = \frac{V_S}{\sqrt{V_n^2}} \leq 0.02 \ll 1$$

The situation is practically equal to that of Case 1 : the signal is **practically invisible** on the oscilloscope display, it’s buried in the noise!
CASE 3: signal $V_s \leq 100 \text{ nV}$ at low frequency $f_s = 10 \text{ Hz}$

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

SIGNAL: the power $P_s = \frac{V_s^2}{2} = 50 \cdot 10^{-16} \text{ V}^2$ is within a bandwidth $\Delta f_s = 1 \text{ Hz}$

so that the effective power density of the signal is $\sqrt{S_s} \approx \sqrt{\frac{P_s}{\Delta f_s}} = 70 \text{nV}/\sqrt{\text{Hz}}$

NOISE: due to the $1/f$ noise, the effective power density at $f_s = 10 \text{ Hz}$ is now **much higher**

$$\sqrt{S_n(f_s)} = \sqrt{S_b + S_b \frac{f_c}{f_s}} = \sqrt{S_b} \sqrt{1 + \frac{f_c}{f_s}} = \sqrt{14,2} \cdot \sqrt{S_b} \approx 71 \text{nV}/\sqrt{\text{Hz}}$$

On the spectrum analyzer display the signal peak is **barely visible, it’s equal to the noise**

$$\frac{\sqrt{S_s}}{\sqrt{S_n(f_s)}} \approx 1$$

**Conclusion: the S/N is insufficient** even with a bandpass filter with narrow bandwidth $\Delta f_b$ matched to the signal $\Delta f_b \approx \Delta f_s$

$$\frac{S}{N} = \sqrt{\frac{P_s}{S_b \Delta f_b}} = \sqrt{\frac{S_s \Delta f_s}{S_n(f_s) \Delta f_b}} \approx \frac{\sqrt{S_s}}{\sqrt{S_n(f_s)}} \leq 1$$
Recovering Narrow-Band Signals

SUMMARY

• For a narrow-band signal plunged in white noise (i.e. with frequency $f_S$ higher than the 1/f noise corner frequency $f_c$) a bandpass filter matched to the signal band is very efficient and makes possible to recover signals even so small that they are buried in the wide-band noise.

• For a narrow-band signal plunged in dominant 1/f noise (i.e. with $f_S$ lower than the 1/f noise corner frequency $f_c$) a bandpass filter matched to the signal is still quite efficient and in many cases makes possible to recover the signal. However, if we consider signals at progressively lower frequency $f_S$, the 1/f noise density at $f_S$ progressively rises, so that the available S/N is progressively reduced.
OPEN QUESTIONS

• We need **efficient band-pass filters** with very narrow band-width. We need to understand how to design and implement such narrow-band filters, but we shall deal with this issue after dealing with the following question.

• If the information is carried by the amplitude of a low-frequency signal, it has to face also 1/f noise. *It would be advantageous to escape this noise by preliminarly transferring the information to a signal at higher frequency.* However:

  a) how can we transfer the signal to higher frequency?

  b) if we transfer to the higher frequency also the 1/f noise that faces the signal, this makes the transfer useless: how can we avoid it?
Recovering Narrow-Band Signals

OPEN QUESTIONS

• For escaping 1/f noise, a low-frequency signal should be transferred to higher frequency before it mixes with 1/f noise of comparable power density: that is, frequency transfer should be done before the stage where the signal meets the 1/f noise source.

• The frequency-transfer stages have their own noise, with different intensity in different types. Unluckily, the types with lowest noise bear other drawbacks, typically a limited capability of transfer, restricted to moderately high frequency.

• For achieving our goal, the signal must be higher than the noise referred to the input of the frequency-transfer stage. If with a given stage the signal is not high enough, preamplifying is not advisable because a preamp brings its 1/f noise. In most cases it is better to transfer the signal «as it is» by means of a frequency transfer stage with lower noise and accept the limitations of this stage, typically a moderate operating frequency.
Moving Signals in Frequency (Signal Modulation)
Amplitude Modulation with DC signal (ideal)

- **Information is brought by** the (VARIABLE) amplitude $A$ of a DC signal $x(t) = A$.
  (NB: a real DC signal is a signal at very low frequency with very narrow bandwidth)

- An analog multiplier circuit combines the signal with a sinusoidal waveform $m(t)$ *(called reference or carrier)* with frequency $f_m$ and **CONSTANT** amplitude $B$

- The information is transferred to the amplitude of a sinusoidal signal $y(t)$ at frequency $f_m$

$$y(t) = A \cdot B \cos(2\pi f_m t + \varphi_m)$$
Amplitude Modulation with DC signal (ideal)

**TIME DOMAIN**

\[ y(t) = x(t) \cdot m(t) = A \cdot B \cos(2\pi f_m t + \phi_m) \]

**FREQUENCY DOMAIN**

\[ Y(f) = X(f) \ast M(f) \]

The signal is shifted in frequency by \( +f_m \) and \(-f_m\) and in phase by \( +\phi_m \) and \(-\phi_m\) respectively.
Convolution in the Frequency Domain

In the **time domain (TD)** the amplitude modulation is the **multiplication**
of the signal $x(t)$ (with variable amplitude $A$)
by the reference waveform $m(t)$ (with standard amplitude $B$)

$$ y(t) = x(t) \cdot m(t) $$

In the **frequency domain (FD)** it is the **convolution**
of the transformed signal $X(f)$ by the transformed reference $M(f)$

$$ Y(f) = X(f) \ast M(f) = \int_{-\infty}^{\infty} X(\alpha) M(f - \alpha) d\alpha $$

Convolution is more complicated in FD than in TD because:

1. the functions to be convolved are twofold, that is, they run in the positive
   and negative sense of the frequency axis
2. **Complex** values must be summed at every frequency for obtaining $Y(f)$.

**In general** the result of FD convolution is not as intuitive as that of TD convolution
and the module $|Y(f)|$ is **NOT** given by the convolution of $|X(f)|$ and $|M(f)|$

$$ |Y(f)| \neq |X(f)| \ast |M(f)| $$

we must first compute the real and imaginary parts of $Y(f)$ and then obtain $|Y(f)|$
Amplitude Modulation with Narrow-Band Signal

In the cases here considered, however, the issue is remarkably simplified because

a) $X(f)$ is confined in a narrow bandwidth $\Delta f_s$

b) $M(f)$ has a line spectrum with (fundamental) frequency $f_m$ that is much greater than the signal bandwidth $f_m >> \Delta f_s$

In the convolution $X(f) \ast M(f)$ each line of $M(f)$ acts on $X(f)$ as follows

- Shifts in frequency every component of $X(f)$ by $+ f_m$ and $- f_m$ (i.e. adds to each frequency $+ f_m$ and $- f_m$)

- Shifts in phase every component of $X(f)$ by $+ \phi_m$ and $- \phi_m$ (i.e. adds to every phase $+ \phi_m$ and $- \phi_m$)

In cases with $\Delta f_s << f_m$, there is no sum of complex numbers to be computed because at any frequency $f$ there is at most one term to be considered, all other terms are negligible.

The result of the convolution is easily visualized: every line of $M(f)$ shifts $X(f)$ in frequency and adds to $X(f)$ its phase. Therefore, $|Y(f)|$ is well approximated by the convolution of $|X(f)|$ and $|M(f)|$ and $|Y(f)|^2$ by the convolution of $|X(f)|^2$ and $|M(f)|^2$

$$ |Y(f)| \approx |X(f)| \ast |M(f)| $$

$$ |Y(f)|^2 \approx |X(f)|^2 \ast |M(f)|^2 $$
Amplitude Modulation with Narrow-Band Signal

Example of quasi-DC NB signal (with very long $T_S$)

$$x(t) = 1(t) \cdot \frac{A}{T_S} e^{-\frac{t}{T_S}}$$

$$m(t) = B \cos(2\pi f_m t + \varphi_m)$$
with $f_m >> 1/T_S$

$$y(t) = x(t) \cdot m(t)$$

FREQUENCY DOMAIN

$$X(f) = A \frac{1}{1 + j2\pi f T_S}$$

$$\Delta f \approx \frac{1}{\pi T_S}$$

$$|X(f)|$$

$$|M(f)|$$

$$|Y(f)|$$

$$Y(f) = X(f) \cdot M(f)$$
Amplitude Modulation with Sinusoidal Signal

\[ x(t) = A \cos(2\pi f_s t) \]

\[ m(t) = B \cos(2\pi f_m t + \varphi_m) \]

with \( f_m \gg f_s \)

|X| \[ \frac{A}{2} \delta(f + f_s) \] \[ \frac{A}{2} \delta(f - f_s) \]

|Y| \[ \frac{AB}{4} \delta(f + f_m + f_s) \] \[ \frac{AB}{4} \delta(f + f_m - f_s) \] \[ \frac{AB}{4} \delta(f - f_m + f_s) \] \[ \frac{AB}{4} \delta(f - f_m - f_s) \]

|M| \[ \frac{B}{2} \delta(f + f_m) \] \[ \frac{B}{2} \delta(f - f_m) \]
Amplitude Modulation with sinusoidal signal

By exploiting the well known trigonometric equation

$$\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)$$

in cases with sinusoidal signal and sinusoidal reference

$$x(t) = A \cos(2\pi f_s t)$$
$$m(t) = B \cos(2\pi f_m t + \phi_m)$$

the result is directly obtained

$$y(t) = x(t) \cdot m(t) = \frac{AB}{2} \cos \left[ 2\pi \left( f_s - f_m \right) t - \phi_m \right] + \frac{AB}{2} \cos \left[ 2\pi \left( f_s + f_m \right) t + \phi_m \right]$$
Squarewave Amplitude Modulation

Modulation with a squarewave reference $m(t)$ can be implemented with circuits based simply on switches and amplifiers, without analog multipliers.

\[ y(t) = x(t) \cdot m(t) \]

- In such cases, the circuit noise referred to the input is due mostly to the switch-devices and is much lower than that of analog multiplier circuits.

- **Metal-contact switches** have the lowest noise, but they can operate at limited switching frequency, typically up to a few 100Hz.

- **Electronic switches** (MOSFET, diodes, etc.) operate up to very high frequencies but have fairly higher noise (anyway MOSFETs operating as switch-device have lower noise than MOSFETs operating as amplifier devices).
Squarewave Amplitude Modulation

Switching example: differential amplifier with alternated input polarity

\[ y(t) = x(t) \cdot m(t) + B \cdot x(t) \]

Switching example: chopper (ON-OFF modulation)

\[ y(t) = x(t) \cdot m(t) \]
**Squarewaves and F-transforms**

\[ m_{sq}(t) = \text{symmetrical squarewave (from +1 to -1) at frequency } f_m \]

\[
B \cdot m_{sq} = \begin{cases} 
B & \text{for } t \in [-5f_m/5, 3f_m/3] \\
B & \text{for } t \in [-3f_m/3, f_m/5] \\
B & \text{for } t \in [f_m/5, 3f_m/3] \\
B & \text{for } t \in [3f_m/3, 5f_m/5]
\end{cases}
\]

\[
B \cdot |M_{sq}| = \begin{cases} 
B & \text{for } f \in [-5f_m/5, 5f_m/5]
\end{cases}
\]

\[
B \cdot M_{sq}(f) = B \cdot \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2} \left[ \delta(f - f_{2k+1}) + \delta(f + f_{2k+1}) \right]
\]

& \text{with } \begin{cases} 
f_{2k+1} = (2k + 1)f_m \\
b_{2k+1} = \frac{(-1)^k}{(2k + 1)} \cdot \frac{4}{\pi}
\end{cases}

In the amplitude modulation:

- each line of the reference M acts like a simple sinusoidal reference, i.e. shifts by its frequency and its phase the signal X and multiplies it by the amplitude \( B \cdot b_{2k+1} \)
- if the squarewave is not perfectly symmetrical (e.g. it has asymmetrical amplitude and/or duration of positive and negative parts) there is also a finite DC component with amplitude \( B_0 \) (possibly very small)
- the DC component does NOT transfer the signal X in frequency, just «amplifies» it by \( B_0 \)
Squarewaves and F-transforms

\[ m_{ch}(t) = \text{chopper squarewave (from +1 to zero) at frequency } f_m \]

Chopper squarewave with amplitude \( B = \frac{B}{\pi} \cdot \frac{1}{5} \)

\[ f_m, -3f_m, -5f_m, \frac{B}{2}, -\frac{B}{2}, \frac{B}{\pi}, 3f_m, 5f_m \]

Symmetrical squarewave with amplitude \( \frac{B}{2} \) + DC component with amplitude \( \frac{B}{2} \)

\[ B \cdot M_{ch}(f) = \frac{B}{2} \cdot M_{sq}(f) + \frac{B}{2} \cdot \delta(f) \]

In the amplitude modulation by a chopper:

- a replica of the signal X «amplified» by \( B/2 \) is transferred in frequency by the squarewave
- another replica of X «amplified» by \( B/2 \) is NOT transferred, it stays where it is
Squarewave Amplitude Modulation

Narrow-band DC input signal

Squarewave reference (slightly asymmetrical case)

Squarewave modulated output signal

\[ |Y| = B \cdot |M_{sq}| \cdot |X| \]

\[ f \]

\[ -5f_m \quad -3f_m \quad -f_m \quad f_m \quad 3f_m \quad 5f_m \]
Summary and Prospect

• As intuitive, narrow-band filtering is very effective for recovering narrow-band signals immersed in wide-band noise.

• Besides wide-band noise, however, other components with power density increasing as the inverse frequency (1/f noise) are ubiquitous in electronic circuitry (amplifiers etc.). In the low-frequency range they are indeed dominant.

• At low frequencies the 1/f noise added by the circuitry is overwhelming, so that the solution of narrow-band filtering becomes progressively less effective and finally insufficient for recovering signals with progressively lower frequency.

• An effective approach to recover a low-frequency signal is to move it to higher frequency before the addition of 1/f noise. That is, to modulate the signal before the circuitry that contains the 1/f noise sources.

• Narrow-band filtering can then be employed to recover the modulated signal; we will now proceed to analyze methods and circuits for narrow band filtering.