COURSE OUTLINE

• Introduction
• Signals and Noise
• Filtering: Band-Pass Filters 1 - BPF1
• Sensors and associated electronics
Band-Pass Filters 1

- Narrow-Band Signals
- Recovering Narrow-Band Signals from Noise
- Moving Signals in Frequency (Signal Modulation)
Narrow-Band Signals
Narrow-Band Signals

Power signals with a narrow power spectrum, that is, a peak with

- center-frequency $f_s$
- bandwidth $\Delta f_s$ which is small in absolute value, typically $\Delta f_s < 10$ Hz, and/or with respect to the center frequency $\Delta f_s \ll f_s$

They approximate well a sinusoid over a wide time interval $T_s \approx 1/\Delta f_s$

**QUESTION**: how can we measure such narrow-band signals in presence of intense white noise? And what if also 1/f noise is present?
Recovering Narrow-Band Signals from Noise
Recovering Narrow-Band Signals from noise

Let’s see some typical examples of signals with

- narrow linewidth $\Delta f_s = 1$ Hz
- small amplitude $V_s \leq 100$ nV

for bringing them to higher level (suitable for processing circuits: filters, meters, etc.) they are amplified by a DC-coupled wide-band preamplifier with

- upper band-limit $f_h = 1 MHz$
- noise spectral density (referred to input) with
  «white» component $\sqrt{S_b} = 5nV/\sqrt{Hz}$
  and $1/f$ component with corner frequency $f_c = 2 kHz$

Let us consider three cases with different center-frequency $f_s$:

- Case 1: **high** frequency $f_s = 100$ kHz
- Case 2: **moderately low** frequency $f_s = 1$ kHz
- Case 3: **low** frequency $f_s = 10$ Hz
Recovering Narrow-Band Signals

**CASE 1:** signal $V_s \leq 100 \: nV$ at high frequency $f_s = 100 \: kHz$

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal to be recovered is at frequency $f_s = 100 \: kHz$ much higher than the noise corner frequency $f_c = 2kHz$, so that we can use a simple high-pass filter with band-limit $f_i = 10kHz$ to cut off the $1/f$ noise and obtain a rms noise (referred to the preamp input)

$$\sqrt{V_n^2} = \sqrt{S_b \cdot (f_h - f_i)} \approx \sqrt{S_b \cdot f_h} = 5 \mu V$$

and therefore

$$\frac{S}{N} = \frac{V_s}{\sqrt{V_n^2}} \leq 0.02 << 1$$

Even the highest signal $V_s = 100 \: nV$ is **practically invisible on the oscilloscope display**! The noise covers a band $\approx 5 \times$ *rms value* $\approx 20 \: \mu V$ and the sinusoidal signal is buried in it!

![Oscilloscope waveform](image)

*Vertical scale 50\(\mu V/\text{div})*

*Horizontal scale 5\(\mu s/\text{div})*
CASE 1: signal $V_s \leq 100 \, nV$ at high frequency $f_s = 100 \, kHz$

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

**SIGNAL:** the power $P_s = \frac{V_s^2}{2} = 50 \cdot 10^{-16} \, V^2$ is within a bandwidth $\Delta f_s = 1 \, Hz$

so that the effective power density of the signal is $\sqrt{S_s} \approx \sqrt{\frac{P_s}{\Delta f_s}} = 70 nV / \sqrt{Hz}$

**NOISE:** the effective power density at $f_s = 100 \, kHz$ is $\sqrt{S_b} = 5 nV / \sqrt{Hz}$

On the spectrum analyzer display the signal peak is very well visible above the noise!

$$\frac{\sqrt{S_s}}{\sqrt{S_b}} = 14 \gg 1$$

**Conclusion:** good S/N can be obtained with a bandpass filter having bandwidth $\Delta f_b$ matched to the signal band $\Delta f_b \approx \Delta f_s$

$$\frac{S}{N} = \sqrt{\frac{P_s}{S_b \Delta f_b}} = \sqrt{\frac{S_s \Delta f_s}{S_b \Delta f_b}} \approx \sqrt{\frac{S_s}{S_b}} = 14 \gg 1$$
CASE 2: signal $V_s \leq 100 \, \text{nV}$ at moderately low frequency $f_s = 1 \, \text{kHz}$

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at $f_s = 1 \, \text{kHz}$ just below the corner frequency $f_c = 2\, \text{kHz}$.

For reducing the 1/f noise we can still use a high-pass filter, but in order to pass the signal, the band-limit $f_i$ must be reduced: $f_i \ll f_s = 1 \, \text{kHz}$, typically $f_i = 100\, \text{Hz}$.

The rms noise referred to the input is

$$
\sqrt{\frac{v_n^2}{n}} \approx \sqrt{S_b (f_h - f_i) + S_b f_c \ln \left( \frac{f_h}{f_i} \right)} \approx \sqrt{S_b f_h + S_b f_c \ln \left( \frac{f_h}{f_i} \right)} \approx \sqrt{S_b f_h} \approx 5 \, \mu V
$$

and therefore

$$
\frac{S}{N} = \frac{V_s}{\sqrt{v_n^2}} \leq 0.02 \ll 1
$$

1/f noise is negligible $S_b f_c \ln \left( \frac{f_h}{f_i} \right) \ll S_b f_h$

The situation is practically equal to that of Case 1: the signal is **practically invisible** on the oscilloscope display, it’s buried in the noise!
Recovering Narrow-Band Signals

CASE 2: signal $V_s \leq 100 \text{ nV}$ at moderately low frequency $f_s = 1 \text{ kHz}$

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

SIGNAL: the power $P_S = \frac{V_s^2}{2} = 50 \cdot 10^{-16} \text{ V}^2$ is within a bandwidth $\Delta f_S = 1 \text{ Hz}$

so that the effective power density of the signal is $\sqrt{S_S} \approx \sqrt{\frac{P_S}{\Delta f_S}} = 70 \text{nV}/\sqrt{\text{Hz}}$

NOISE: due to the $1/f$ noise, the effective power density at $f_s = 1 \text{ kHz}$ is somewhat higher

$$\sqrt{S_n(f_s)} = \sqrt{S_b + S_b \frac{f_c}{f_s}} = \sqrt{S_b} \sqrt{1 + \frac{f_c}{f_s}} = \sqrt{3} \cdot \sqrt{S_b} \approx 8,7 \text{nV}/\sqrt{\text{Hz}}$$

Anyway, on the spectrum analyzer display the signal peak is still well visible above the noise

$$\frac{\sqrt{S_S}}{\sqrt{S_n(f_s)}} = 8 > 1$$

Conclusion: a bandpass filter with bandwidth $\Delta f_b$ matched to the signal $\Delta f_b \approx \Delta f_S$ still gives a fairly good $S/N$
Recovering Narrow-Band Signals

**CASE 3: signal \( V_s \leq 100 \text{ nV} \) at low frequency \( f_s = 10 \text{ Hz} \)

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at \( f_s = 10 \text{ Hz} \) much below the corner frequency \( f_c = 2 \text{kHz} \).

For reducing the the 1/f noise we can still use a high-pass filter, but with strongly reduced band-limit \( f_i \ll f_s = 10 \text{ Hz} \), typically \( f_i = 1 \text{ Hz} \). The rms noise referred to input is

\[
\sqrt{\frac{V_n^2}{N}} \approx \sqrt{S_b \left( f_h - f_i \right) + S_b f_c \ln \left( \frac{f_h}{f_i} \right)} \approx \sqrt{S_b f_h + S_b f_c \ln \left( \frac{f_h}{f_i} \right)} \approx \sqrt{S_b f_h} \approx 5 \mu\text{V}
\]

1/f noise is negligible \( S_b f_c \ln \left( \frac{f_h}{f_i} \right) \ll S_b f_h \)

and therefore \( \frac{S}{N} = \frac{V_s}{\sqrt{V_n^2}} \leq 0.02 \ll 1 \)

The situation is practically equal to that of Case 1: the signal is **practically invisible on the oscilloscope display, it’s buried in the noise!**
Recovering Narrow-Band Signals

**CASE 3: signal** $V_s \leq 100 \text{ nV}$ **at low frequency** $f_s = 10 \text{ Hz}$

b) observing the power spectrum in **FREQUENCY DOMAIN**, i.e. on spectrum analyzer display

**SIGNAL:** the power $P_s = \frac{V_s^2}{2} = 50 \cdot 10^{-16} \text{ V}^2$ is within a bandwidth $\Delta f_s = 1 \text{ Hz}$ so that the effective power density of the signal is $\sqrt{S_s} \approx \sqrt{\frac{P_s}{\Delta f_s}} = 70 \text{nV}/\sqrt{\text{Hz}}$

**NOISE:** due to the 1/f noise, the effective power density at $f_s = 10 \text{ Hz}$ is now much higher

$$\sqrt{S_n(f_s)} = \sqrt{S_b + S_b \frac{f_c}{f_s}} = \sqrt{S_b \sqrt{1 + \frac{f_c}{f_s}}} = \sqrt{14} \cdot 2 \cdot \sqrt{S_b} \approx 71 \text{nV}/\sqrt{\text{Hz}}$$

On the spectrum analyzer display the signal peak is barely visible, it’s equal to the noise

$$\frac{\sqrt{S_s}}{\sqrt{S_n(f_s)}} \approx 1$$

**Conclusion:** the **S/N is insufficient** even with a bandpass filter with narrow bandwidth $\Delta f_b \approx \Delta f_s$

$$\frac{S}{N} = \sqrt{\frac{P_s}{S_b \Delta f_b}} = \sqrt{\frac{S_s \Delta f_s}{S_n(f_s) \Delta f_b}} \approx \frac{\sqrt{S_s}}{\sqrt{S_n(f_s)}} \leq 1$$
SUMMARY

- For a narrow-band signal plunged in white noise (i.e. with frequency $f_S$ higher than the 1/f noise corner frequency $f_c$) a bandpass filter matched to the signal band is very efficient and makes possible to recover signals even so small that they are buried in the wide-band noise.

- For a narrow-band signal plunged in dominant 1/f noise (i.e. with $f_S$ lower than the 1/f noise corner frequency $f_c$) a bandpass filter matched to the signal is still quite efficient and in many cases makes possible to recover the signal. However, if we consider signals at progressively lower frequency $f_S$, the 1/f noise density at $f_S$ progressively rises, so that the available S/N is progressively reduced.
OPEN QUESTIONS

• We need **efficient band-pass filters** with very narrow band-width. We need to understand how to design and implement such narrow-band filters, but we shall deal with this issue after dealing with the following question.

• If the information is carried by the amplitude of a low-frequency signal, it has to face also 1/f noise. *It would be advantageous to escape this noise by preliminarly transferring the information to a signal at higher frequency.* However:

  a) how can we transfer the signal to higher frequency?

  b) if we transfer to the higher frequency also the 1/f noise that faces the signal, this makes the transfer useless: how can we avoid it?
OPEN QUESTIONS

• For escaping 1/f noise, a low-frequency signal should be transferred to higher frequency before it mixes with 1/f noise of comparable power density: that is, frequency transfer should be done before the stage where the signal meets the 1/f noise source.

• The frequency-transfer stages have their own noise, with different intensity in different types. Unluckily, the types with lowest noise bear other drawbacks, typically a limited capability of transfer, restricted to moderately high frequency.

• For achieving our goal, the signal must be higher than the noise referred to the input of the frequency-transfer stage. If with a given stage the signal is not high enough, preamplifying is not advisable because a preamp brings its 1/f noise. In most cases it is better to transfer the signal «as it is» by means of a frequency transfer stage with lower noise and accept the limitations of this stage, typically a moderate operating frequency.
Moving Signals in Frequency
(Signal Modulation)
Amplitude Modulation with DC signal (ideal)

- **Information is brought by** the (VARIABLE) amplitude $A$ of a DC signal $x(t) = A$.
  (NB: a real DC signal is a signal at very low frequency with very narrow bandwidth)

- An analog multiplier circuit combines the signal with a sinusoidal waveform $m(t)$
  *(called reference or carrier)* with frequency $f_m$ and **CONSTANT** amplitude $B$

- The information is transferred to the amplitude of a sinusoidal signal $y(t)$ at frequency $f_m$
  $$y(t) = A \cdot B \cos(2\pi f_m t + \varphi_m)$$
Amplitude Modulation with DC signal (ideal)

**TIME DOMAIN**

\[ x(t) \]
\[ m(t) \]
\[ y(t) = x(t) \cdot m(t) = A \cdot B \cos(2\pi f_m t + \varphi_m) \]

**FREQUENCY DOMAIN**

\[ |X| = A \delta(f) \]
\[ |M| = \frac{B}{2} \delta(f + f_m) \]
\[ |Y| = \frac{A \cdot B}{2} \delta(f + f_m) \]
\[ \frac{A \cdot B}{2} \delta(f - f_m) \]

\[ Y(f) = X(f) \cdot M(f) \]

The signal is shifted in frequency by \(+f_m\) and \(−f_m\) and in phase by \(+\varphi_m\) and \(−\varphi_m\) respectively.
Convolution in the Frequency Domain

In the **time domain (TD)** the amplitude modulation is the **multiplication**

of the signal $x(t)$ (with variable amplitude $A$)

by the reference waveform $m(t)$ (with standard amplitude $B$)

$$y(t) = x(t) \cdot m(t)$$

In the **frequency domain (FD)** it is the **convolution**

of the transformed signal $X(f)$ by the transformed reference $M(f)$

$$Y(f) = X(f) \ast M(f) = \int_{-\infty}^{\infty} X(\alpha) M(f - \alpha) d\alpha$$

Convolution is more complicated in FD than in TD because:

1. the functions to be convolved are twofold, that is, they run in the positive and negative sense of the frequency axis

2. **Complex** values must be summed at every frequency for obtaining $Y(f)$.

**In general** the result of FD convolution is not as intuitive as that of TD convolution and the module $|Y(f)|$ is **NOT** given by the convolution of $|X(f)|$ and $|M(f)|$

$$|Y(f)| \neq |X(f)| \ast |M(f)|$$

we must first compute the real and imaginary parts of $Y(f)$ and then obtain $|Y(f)|$
Amplitude Modulation with Narrow-Band Signal

In the cases here considered, however, the issue is remarkably simplified because

a) \( X(f) \) is confined in a **narrow** bandwidth \( \Delta f_s \)

b) \( M(f) \) has a line spectrum with (fundamental) frequency \( f_m \) that is much greater than the signal bandwidth \( f_m >> \Delta f_s \)

In the convolution \( X(f) \ast M(f) \) each line of \( M(f) \) acts on \( X(f) \) as follows

- Shifts in frequency every component of \( X(f) \) by \(+ f_m \) and \(- f_m \)
  (i.e. adds to each frequency \(+ f_m \) and \(- f_m \) )

- Shifts in phase every component of \( X(f) \) by \(+ \varphi_m \) and \(- \varphi_m \)
  (i.e. adds to every phase \(+ \varphi_m \) and \(- \varphi_m \) )

In cases with \( \Delta f_s << f_m \), there is **no sum of complex numbers** to be computed because at any frequency \( f \) there is at most one term to be considered, all other terms are negligible.

The result of the convolution is easily visualized: every line of \( M(f) \) shifts \( X(f) \) in frequency and adds to \( X(f) \) its phase. Therefore, \( |Y(f)| \) is well approximated by the convolution of \( |X(f)| \) and \( |M(f)| \) and \( |Y(f)|^2 \) by the convolution of \( |X(f)|^2 \) and \( |M(f)|^2 \)

\[
|Y(f)| \approx |X(f)| \ast |M(f)| \quad |Y(f)|^2 \approx |X(f)|^2 \ast |M(f)|^2
\]
Amplitude Modulation with Narrow-Band Signal

Example of quasi-DC NB signal (with very long $T_s$)

$$x(t) = 1(t) \cdot \frac{A}{T_s} e^{-\frac{t}{T_s}}$$

$$m(t) = B \cos(2\pi f_m t + \varphi_m)$$

with $f_m >> 1/T_s$

$$y(t) = x(t) \cdot m(t)$$

**FREQUENCY DOMAIN**

$$X(f) = A \frac{1}{1 + j2\pi f T_s}$$

$$\Delta f \approx \frac{1}{\pi T_s}$$

$$|X|$$

$$|M|$$

$$|Y|$$

$$Y(f) = X(f) \cdot M(f)$$
Amplitude Modulation with Sinusoidal Signal

\[ x(t) = A \cos(2\pi f_s t) \]
\[ m(t) = B \cos(2\pi f_m t + \phi_m) \]

with \( f_m \gg f_s \)
Amplitude Modulation with sinusoidal signal

By exploiting the a well known trigonometric equation

\[
\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha - \beta) + \frac{1}{2} \cos(\alpha + \beta)
\]

in cases with sinusoidal signal and sinusoidal reference

\[
x(t) = A \cos(2\pi f_s t)
\]

\[
m(t) = B \cos(2\pi f_m t + \phi_m)
\]

the result is directly obtained

\[
y(t) = x(t) \cdot m(t) = \frac{AB}{2} \cos \left[ 2\pi \left( f_s - f_m \right) t - \phi_m \right] + \frac{AB}{2} \cos \left[ 2\pi \left( f_s + f_m \right) t + \phi_m \right]
\]
Squarewave Amplitude Modulation

Modulation with a squarewave reference \( m(t) \) can be implemented with circuits based simply on switches and amplifiers, without analog multipliers.

\[
y(t) = x(t) \cdot m(t)
\]

- In such cases, the circuit noise referred to the input is due mostly to the switch-devices and is much lower than that of analog multiplier circuits.
- **Metal-contact switches** have the lowest noise, but they can operate at limited switching frequency, typically up to a few 100Hz.
- **Electronic switches** (MOSFET, diodes, etc.) operate up to very high frequencies but have fairly higher noise (anyway MOSFETs operating as switch-device have lower noise than MOSFETs operating as amplifier devices).
Squarewave Amplitude Modulation

Switching example: differential amplifier with alternated input polarity

$$y(t) = x(t) \cdot m(t) + B$$

Switching example: chopper (ON-OFF modulation)

$$y(t) = x(t) \cdot m(t)$$
**Squarewaves and F-transforms**

$m_{sq}(t) = \text{symmetrical squarewave (from +1 to -1) at frequency } f_m$

\[
B \cdot m_{sq}(t) = B \cdot \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2} \left[ \delta(f - f_{2k+1}) + \delta(f + f_{2k+1}) \right]
\]

In the amplitude modulation:

- each line of the reference $M$ acts like a simple sinusoidal reference, i.e. shifts by its frequency and its phase the signal $X$ and multiplies it by the amplitude $B \cdot b_{2k+1}$

- if the squarewave is **not perfectly symmetrical** (e.g. it has asymmetrical amplitude and/or duration of positive and negative parts) there is also a finite **DC component with amplitude $B_0$ (possibly very small)**

- the DC component does NOT transfer the signal $X$ in frequency, just «amplifies» it by $B_0$

\[
\begin{align*}
B \cdot M_{sq}(f) &= B \cdot \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2} \left[ \delta(f - f_{2k+1}) + \delta(f + f_{2k+1}) \right] \\
&= B \cdot \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2} \left[ \delta(f - f_{2k+1}) + \delta(f + f_{2k+1}) \right]
\end{align*}
\]

\[
\begin{align*}
&\text{with } \begin{cases} 
  f_{2k+1} = (2k + 1)f_m \\
  b_{2k+1} = \frac{(-1)^k}{(2k+1)} \cdot \frac{4}{\pi}
\end{cases}
\end{align*}
\]
Squarewaves and F-transforms

\[ m_{ch}(t) = \text{chopper squarewave (from +1 to zero) at frequency } f_m \]

Chopper squarewave with amplitude \( B = \)
\[ = \text{Symmetrical squarewave with amplitude } B/2 + \text{ DC component with amplitude } B/2 \]

\[ B \cdot M_{ch}(f) = \frac{B}{2} \cdot M_{sq}(f) + \frac{B}{2} \cdot \delta(f) \]

In the amplitude modulation by a chopper:
- a replica of the signal \( X \) «amplified» by \( B/2 \) is transferred in frequency by the squarewave
- another replica of \( X \) «amplified» by \( B/2 \) is NOT transferred, it stays where it is
Squarewave Amplitude Modulation

Narrow-band DC input signal

Squarewave reference (slightly asymmetrical case)

Squarewave modulated output signal

\[ |Y| = B \cdot |M_{sq}| \cdot |X| \]

\[ f = -5f_m, -3f_m, -f_m, f_m, 3f_m, 5f_m \]
Summary and Prospect

• As intuitive, narrow-band filtering is very effective for recovering narrow-band signals immersed in wide-band noise.

• Besides wide-band noise, however, other components with power density increasing as the inverse frequency (1/f noise) are ubiquitous in electronic circuitry (amplifiers etc.). In the low-frequency range they are indeed dominant.

• At low frequencies the 1/f noise added by the circuitry is overwhelming, so that the solution of narrow-band filtering becomes progressively less effective and finally insufficient for recovering signals with progressively lower frequency.

• An effective approach to recover a low-frequency signal is to move it to higher frequency before the addition of 1/f noise. That is, to modulate the signal before the circuitry that contains the 1/f noise sources.

• Narrow-band filtering can then be employed to recover the modulated signal; we will now proceed to analyze methods and circuits for narrow band filtering.