# COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering: OPF1 Optimum Filtering 1
- Sensors and associated electronics

# Optimum Filtering 1



- Amplitude measurement of pulse signals
- Filtering for amplitude measurement of signals in stationary white noise
- Optimum filtering with white stationary noise: matched filter

# Amplitude Measurement of Pulse Signals

# Pulse-Amplitude Measurements



**Pulse signals** can carry information, which in many cases is contained in the **amplitude** of the pulse, not in the pulse shape or in other parameters (e.g. pulse risetime, duration, etc). These cases can be better illustrated by a couple of examples

- **Automated analysis of biological cells**
  Fluorescence methods are often employed. A cells in diluted solution are labeled with a fluorescent dye that attaches specifically to a given component of the cell. The cells are conveyed by a laminar stream in a small duct and cross a laser beam that excites their fluorescence. The fluorescence pulse emitted by a cell has **intensity** proportional to the **quantity of component** in the cell. By measuring and classifying many pulses (i.e. by collecting the measurement histogram) the distribution of the component in the cell population is obtained.

- **Ionizing radiation spectrometry**
  The radiation detectors generate for each quantum of radiation received (e.g. a Gamma ray) a current pulse with **charge** proportional to the quantum **energy**. By measuring the charge of each pulse and collecting the histogram of measurements, the radiation distribution in energy is obtained (the energy spectrum). It is thus possible to identify radionuclides in the source (e.g. Plutonium in the elements of a nucler reactor); to measure their quantity; to monitor radiation doses etc.

# Ionizing Radiation Spectrometry



It is a typical case of pulse-amplitude measurement , let us consider it in more detail

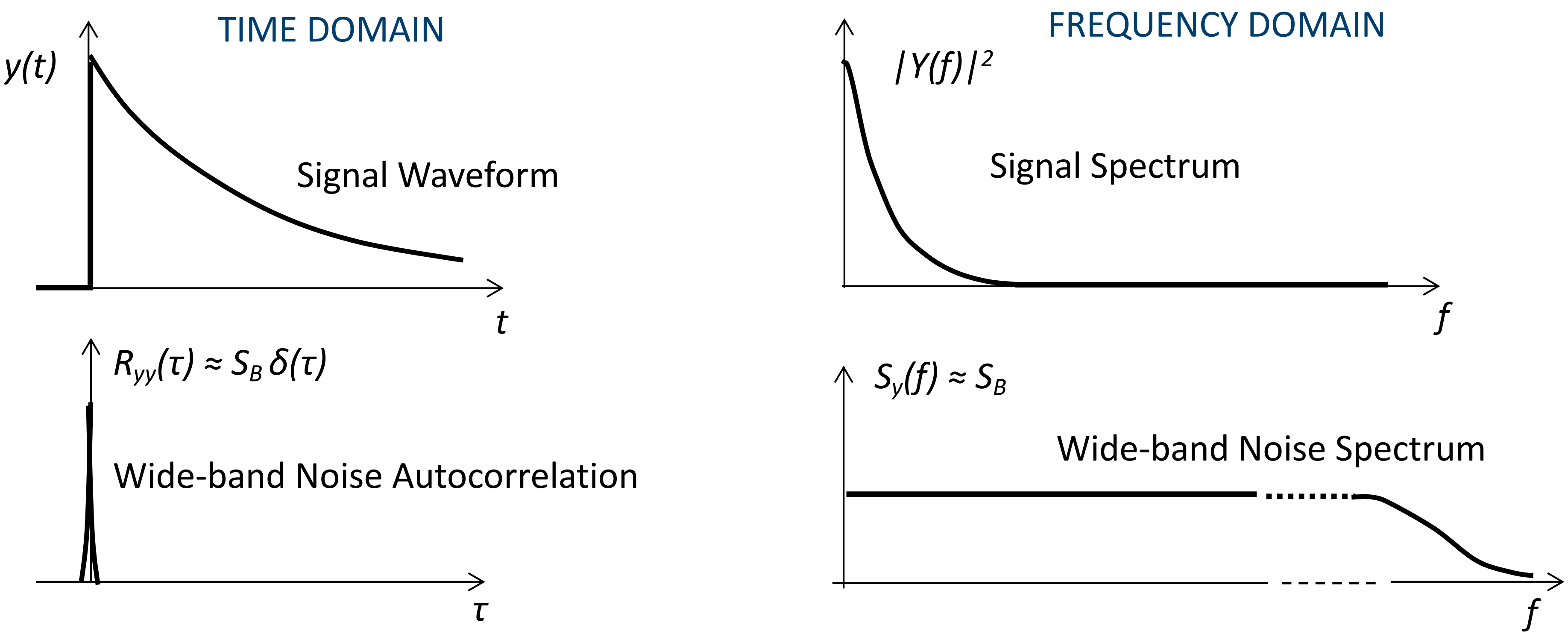
- Detectors of ionizing radiation (e.g. Gamma rays) generate a current pulse signal for each radiation quantum received
- The **charge** of the pulse signal is proportional to the radiation quantum **energy**
- **The electronics has to measure the charge of each pulse**, not its shape or position in time. In fact, the initial time of each pulse is known (signaled by auxiliary electronics) and all pulses have equal shape, i.e. equal waveform with normalized amplitude
- **The precision of the measure is limited by noise sources in the detector** and in the electronics
- Measurements of many pulses are collected and **classified by size**
- **The distribution of the measured pulse-charge reflects the distribution in energy** (spectrum) of the radiation. The energy scale can be calibrated by measuring radiations with known energy.
- The energy spectrum of the radiation **gives information about the radioactive source** (type and quantity of radionuclides, etc.)

# Filtering for Amplitude Measurement of Signals with White Stationary Noise

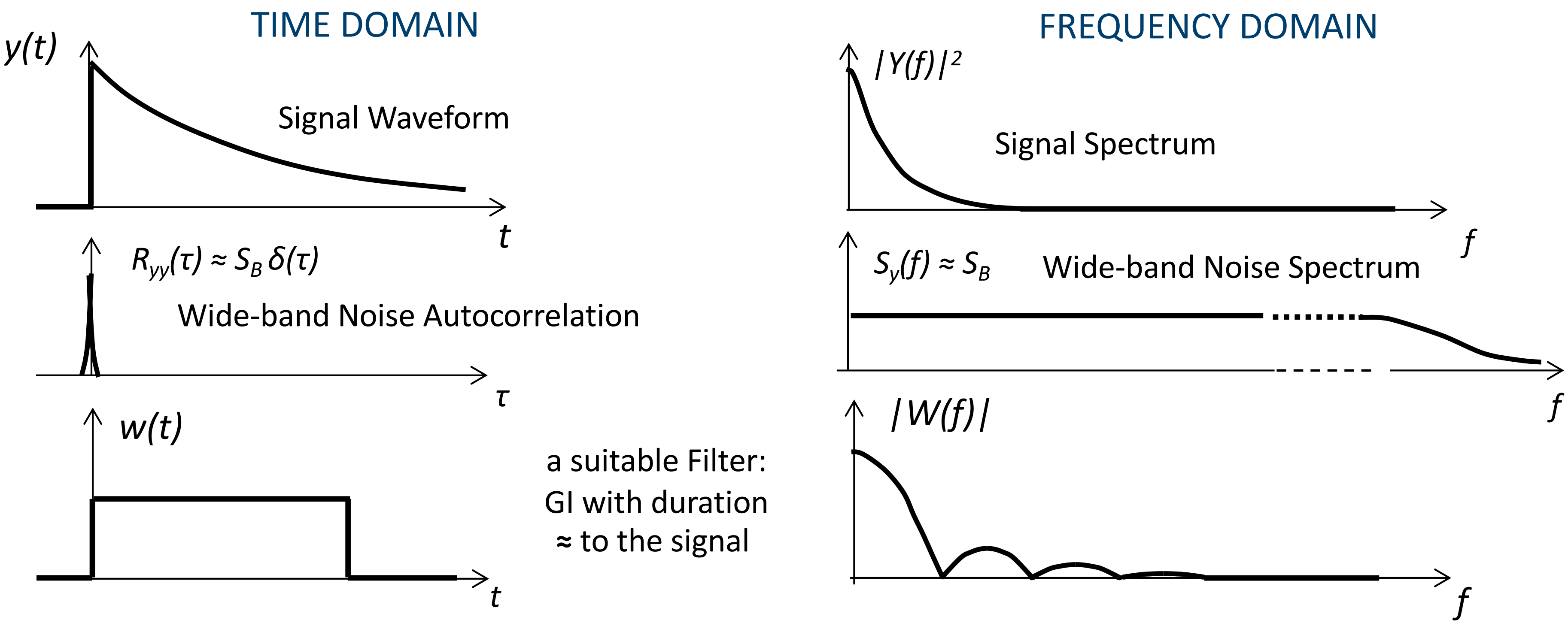
# Filtering to measure Amplitude of Signals beset by White Stationary Noise

Let's consider first a basic case: **pulse signals** accompanied by **stationary white noise**.
This ideal case is a good approximation for **real cases** where pulse signals are accompanied by wide-band noise, i.e. noise with

- Narrow autocorrelation, i.e. **width much smaller than the signal duration**
- Wideband uniform spectrum, i.e. **upper bandlimit much higher than that of the signal**



# Filtering to measure Amplitude of Signals in White Noise




**For a measurement of pulse amplitude, a filter has to collect most of the signal and reject most of the noise. It's intuitive that its action should be:**

- as seen in time, more or less to average the signal and the white noise over the time interval occupied by the signal
- as seen in frequency, more or less to pass the low frequency range occupied by the signal and cut the higher frequency range where only white noise is present

This means that it's a **low-pass filter (LPF) tailored to the signal** (see GI example above)

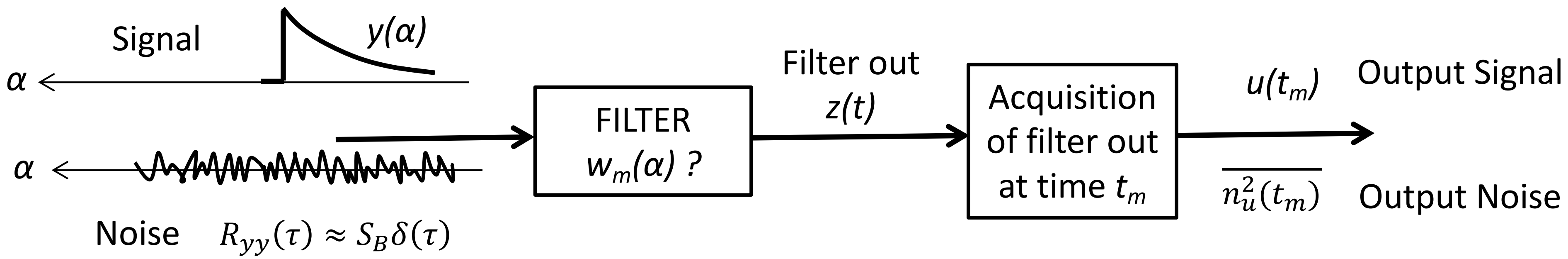
# Filtering to measure Amplitude of Signals in White Noise

**Low-pass filters tailored to the signal** are suitable, but we'd like to know more, since basic questions are still open:

- *is there an optimal filter and if yes, what is it?*
- *If yes, what is the **best obtainable result**? That is, what is the optimized S/N and what is the smallest measurable amplitude?*

The issue is to find out the optimal weighting function, since it completely characterizes a linear filter.
Let's set in evidence the signal area A and the normalized waveform *b(t)*

$y(t) = A \cdot b(t)$ with $\int_{-\infty}^{\infty} b(t)\,dt = 1$




**QUESTION**: is there a weighting function $w_m(\alpha)$ that optimizes $\frac{S}{N} = \frac{u(t_m)}{\sqrt{\overline{n_u^2(t_m)}}}$ ?

# Optimum Filtering with White Stationary Noise

# Optimum Filtering of Signals in White Noise



The signal and noise acquired in the measurement are

$$u(t_m) = \int_{-\infty}^{\infty} y(\alpha)\, w_m(\alpha) d\alpha = A \cdot \int_{-\infty}^{\infty} b(\alpha)\, w_m(\alpha) d\alpha = A \cdot k_{bw}(0)$$

$$\overline{n_u^2(t_m)} = \int_{-\infty}^{\infty} R_{yy}(\alpha)\;\; k_{ww}(\alpha) d\alpha = S_B \cdot \int_{-\infty}^{\infty} w_m^2\,(\alpha) d\alpha = S_B \cdot k_{ww}(0)$$

Therefore

$$\left(\frac{S}{N}\right)^2 = \frac{u^2(t_m)}{\overline{n_u^2(t_m)}} = \frac{A^2}{S_B} \cdot \frac{k_{bw}^2(0)}{k_{ww}(0)}$$

The $w_m(\alpha)$ that optimizes S/N for a given pulse shape $b(\alpha)$ is found by exploiting the known property of correlation functions (based on Schwartz's inequality)

$$k_{bw}^2(0) \leq k_{bb}(0) \cdot k_{ww}(0)$$ that is $$\frac{k_{bw}^2(0)}{k_{ww}(0)} \leq k_{bb}(0)$$

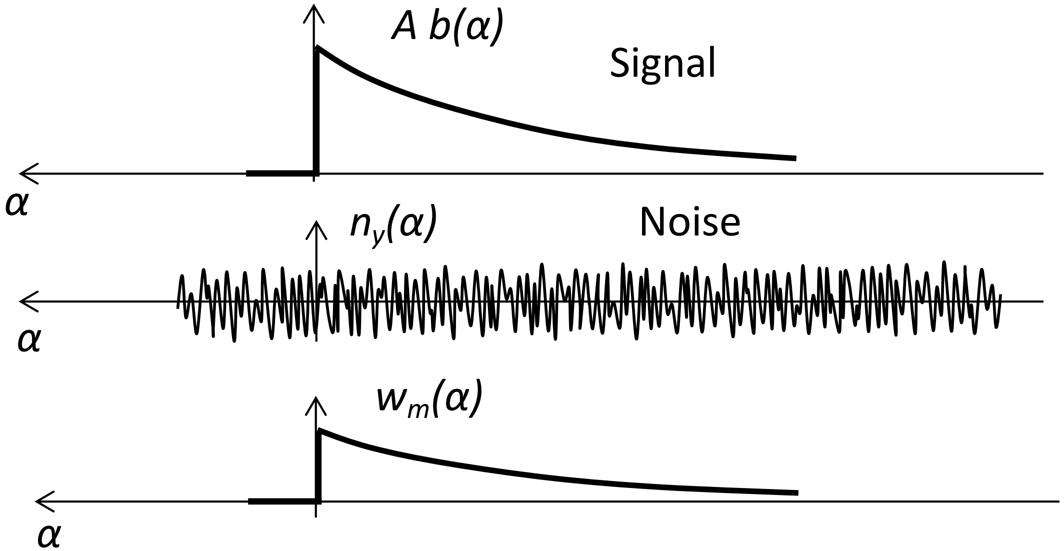
where the maximum is achieved with **filter weighting proportional to the signal shape**

$w_m(\alpha) \propto b(\alpha)$ which normalized to unit area is $\boxed{w_m(\alpha) = b(\alpha)}$

and gives $\max\left[k_{bw}^2(0)\right] = k_{bb}^2(0)$ that is $\max\left[\frac{k_{bw}^2(0)}{k_{ww}(0)}\right] = k_{bb}(0)$

# Matched Filter






The best result in measurements of the amplitude of signal pulses accompanied by **stationary** white noise is obtained with weighting function equal to the signal shape. This conclusion is intuitive: since the noise is uncorrelated, the output noise power is the weighted sum of the noise instantaneous power at all times; since this power is equal at all times, it is convenient to give higher weight when the signal is higher.

The filter with weighting function $w_m(\alpha)$ matched to the signal shape $b(\alpha)$

$$w_m(\alpha) \propto b(\alpha)$$

is indeed called **MATCHED FILTER**

# Optimum S/N for Signals in White Noise

**The optimum S/N provided by the Matched Filter is**

$$\left(\frac{S}{N}\right)^2_{opt} = \frac{A^2}{S_B} \cdot k_{bb}(0) = \frac{A^2}{S_B} \cdot \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha$$


recalling that the energy $E_y$ of the signal *A b(t)* is

$$E_y = A^2 \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha = A^2 \int_{-\infty}^{\infty} B^2(f)\, df$$

we see that $(S/N)^2_{opt}$ is simply

$$\left(\frac{S}{N}\right)^2_{opt} = \frac{E_y}{S_B} = \frac{signal\ energy}{noise\ power\ density\ (bilateral)}$$

# Minimal Measurable Amplitude of Signals in White Noise



By considering explicitly the energy of the normalized signal *b(t)*

$$E_b = \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha = \int_{-\infty}^{\infty} B^2(f)\, df$$

we set in evidence the signal amplitude *A* (the signal area)

$$\left(\frac{S}{N}\right)_{opt}^2 = A^2 \cdot \frac{E_b}{S_B}$$

The minimum measurable amplitude $A_{min}$ is defined as the amplitude that gives $(S/N)_{opt} = 1$, therefore it is

$$A_{min} = \frac{\sqrt{S_B}}{\sqrt{E_b}} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha}} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} B^2(f)\, df}}$$

that is $A_{\min}^2$ = (spectral density of noise) / (energy of the normalized signal)

# Optimum Filtering with any Stationary Noise

# Optimum Filtering with Any Stationary Noise



- Optimal filters for measuring the amplitude of pulse signals accompanied by a **any stationary noise** can be obtained by an **extension of the solution for white noise**.

- We can begin by processing the signal and noise with a noise-whitening filter that transforms the noise in white noise by a reversible transformation. In fact:
  a) for a given stationary noise, it is always possible to find a **constant-parameter** linear filter that produces such a result, since it has transfer function $H_b(f)$ such that

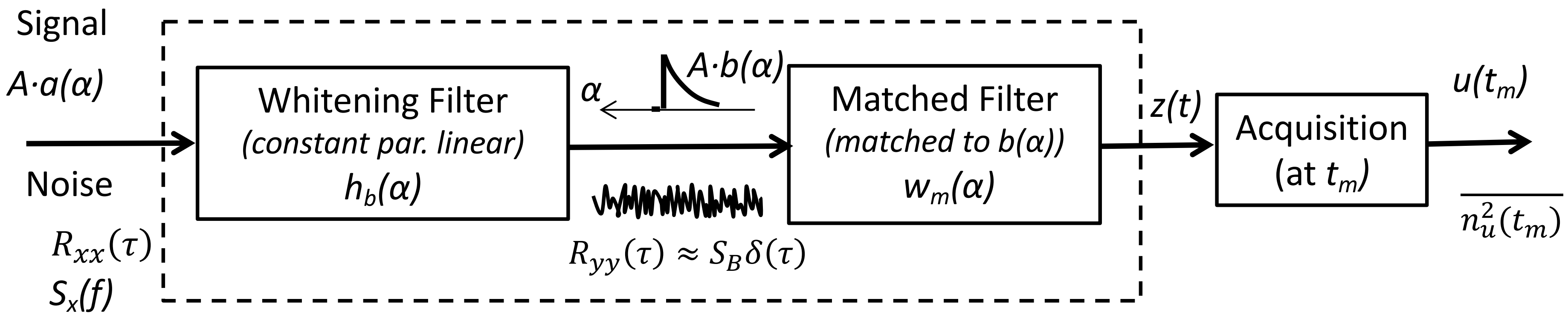
$$|H_b(f)|^2 \propto \frac{1}{S_x(f)}$$

  b) the transformations performed by constant-parameter linear filters are reversible.

- **Reversibility is essential**: nothing is lost in the transformation and whatever is done by the whitening filter can be reversed by the following filters. We can then proceed towards the optimum, since we know what to do in the situation at the output of the whitening filter: we have pulse signals accompanied by white stationary noise and we know that a matched filter performs the optimum filtering.

- **In conclusion**, the optimum weighting function for measuring the amplitude of pulses with any stationary noise is obtained as the **overall weighting function of two cascaded filter** stages: a **whitening** filter followed by a **matched** filter.

# Optimum Filtering with Any Stationary Noise




- The whitening filter modifies the waveform of the signal, hence the following filter is matched to this modified signal, not to the input signal!
- The subdivision of the optimum filter in whitening filter and matched filter is a useful theoretical approach for analyzing the problem and finding the overall optimal weighting, but it is **NOT THE NECESSARY STRUCTURE** of the the optimum filter.
- In principle, we find the optimal weighting by combining whitening filter and matched filter. In practice, we can implement this optimal weighting with different filter structures employing any kind of linear filter (constant or time-variant parameters; passive or active; etc.)
- The wide liberty in the implementation of the optimum filter is very important, since in many cases it is quite difficult to design the noise-whitening filter and even more difficult to implement it, due to practical limitations of the real components (limited linear dynamic range; noise in the circuit elements; etc.).

# About the Noise Whitening Filter



- For a given noise with spectrum $S_x(f)$ the **whitening** filter is a **constant-parameter linear** filter that has transfer function $H_b(f)$ such that $|H_b(f)|^2 \propto 1/S_x(f)$ (in time domain: filter autocorrelation function $k_{bb}(\tau)$ such that the convolution with the noise autocorrelation $R_{xx}(\tau)$ produces a $\delta$-like autocorrelation $R_{yy}(\tau) \propto \delta(\tau)$ )
- The action of the whitening filter is more evident in cases where the actual noise results from white noise filtered by some circuit. For example, consider the Johnson noise of a resistor passed through an amplifier with upper band-limit set by a simple pole at low frequency. The whitening filter simply reverts the filtering by the amplifier with a transformation that cancels the low-pass pole.