

COURSE OUTLINE

- Introduction
- Signals and Noise
- Filtering: OPF1 Optimum Filtering 1
- Sensors and associated electronics

- Amplitude measurement of pulse signals
- Filtering for amplitude measurement of signals in stationary white noise
- Optimum filtering with white stationary noise: matched filter

Amplitude Measurement of Pulse Signals

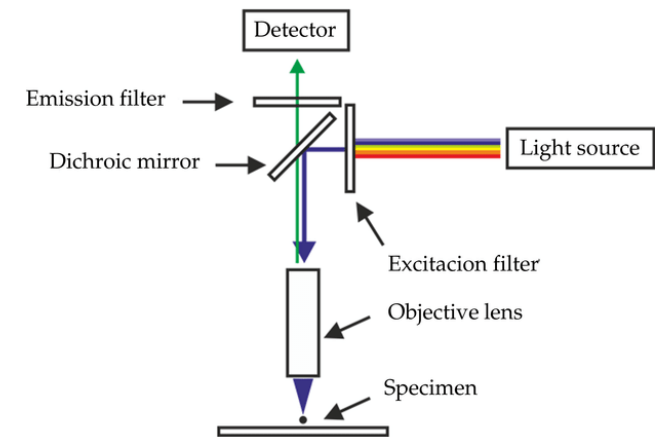
Pulse signals can carry information, which in many cases is contained in the **amplitude** of the pulse (or in the **area**), not in the pulse shape or in other parameters (e.g. pulse risetime, duration, etc). These cases can be better illustrated by a couple of examples

- **Automated analysis of biological cells**

Fluorescence methods are often employed. Cells are labeled with a fluorescent dye that attaches specifically to a given component of the cell. A laser beam excites their fluorescence. The fluorescence pulse emitted by a cell has **intensity (amplitude)** proportional to the **quantity of component** in the cell. By measuring and classifying many pulses (i.e. by collecting the measurement histogram) the distribution of the component in the cell population is obtained.

- **Ionizing radiation spectrometry**

The radiation detectors generate for each quantum of radiation received (e.g. a Gamma ray) a current pulse with **charge (area)** proportional to the quantum **energy**. By collecting the histogram of measurements, the energy spectrum is obtained. It is thus possible, for example, to identify radionuclides in the source e.g. Plutonium in the elements of a nuclear reactor.



It is a typical case of pulse-amplitude measurement , let us consider it in more detail

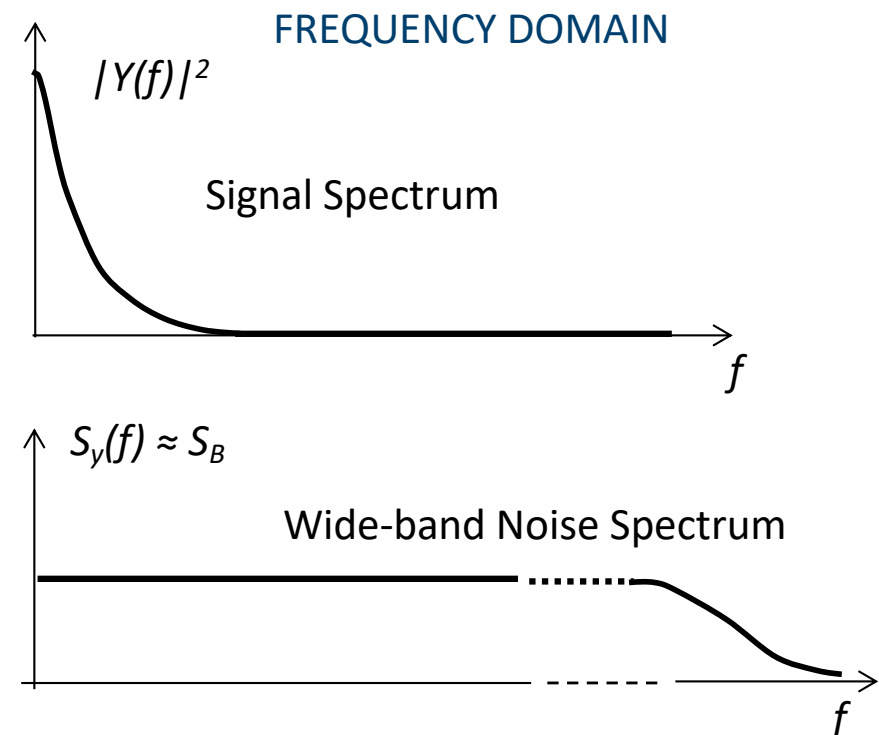
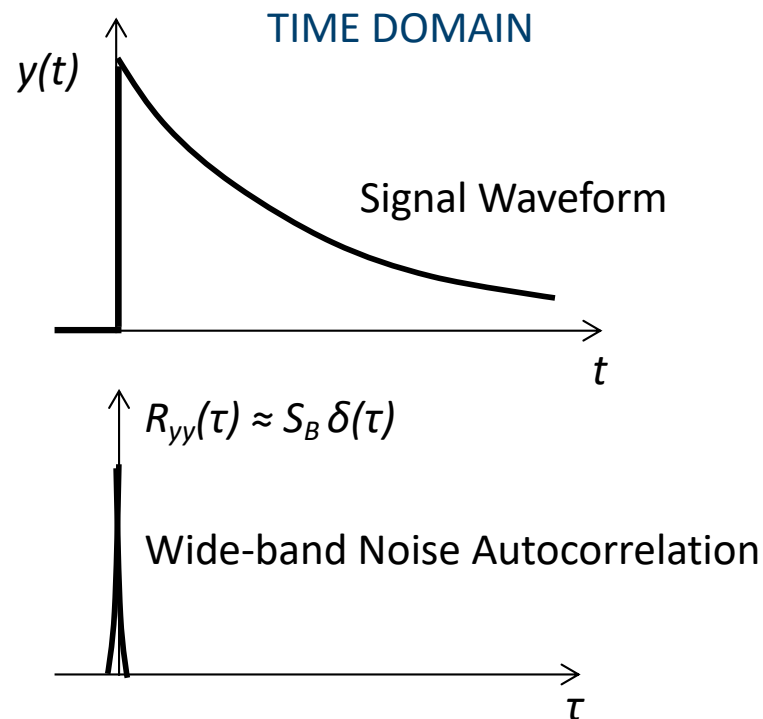
- Detectors of ionizing radiation (e.g. Gamma rays) generate a current pulse signal for each radiation quantum received
- The **charge (area)** of the pulse signal is proportional to the radiation quantum **energy**
- **The electronics has to measure the charge of each pulse**, not its shape or position in time. In fact, the initial time of each pulse is known (signaled by auxiliary electronics) and all pulses have **equal shape**
- As soon as the shape is fixed, **Amplitude** and **Area** give us the same information, since one can be derived from the other
- **The precision of the measure is limited by noise sources in the detector** and in the electronics
- Measurements of many pulses are collected and **classified by size**
- **The distribution of the measured pulse-charge reflects the distribution in energy** (spectrum) of the radiation.
- The energy spectrum of the radiation **gives information about the radioactive source** (type and quantity of radionuclides, etc.)

Filtering for Amplitude Measurement of Signals with White Stationary Noise

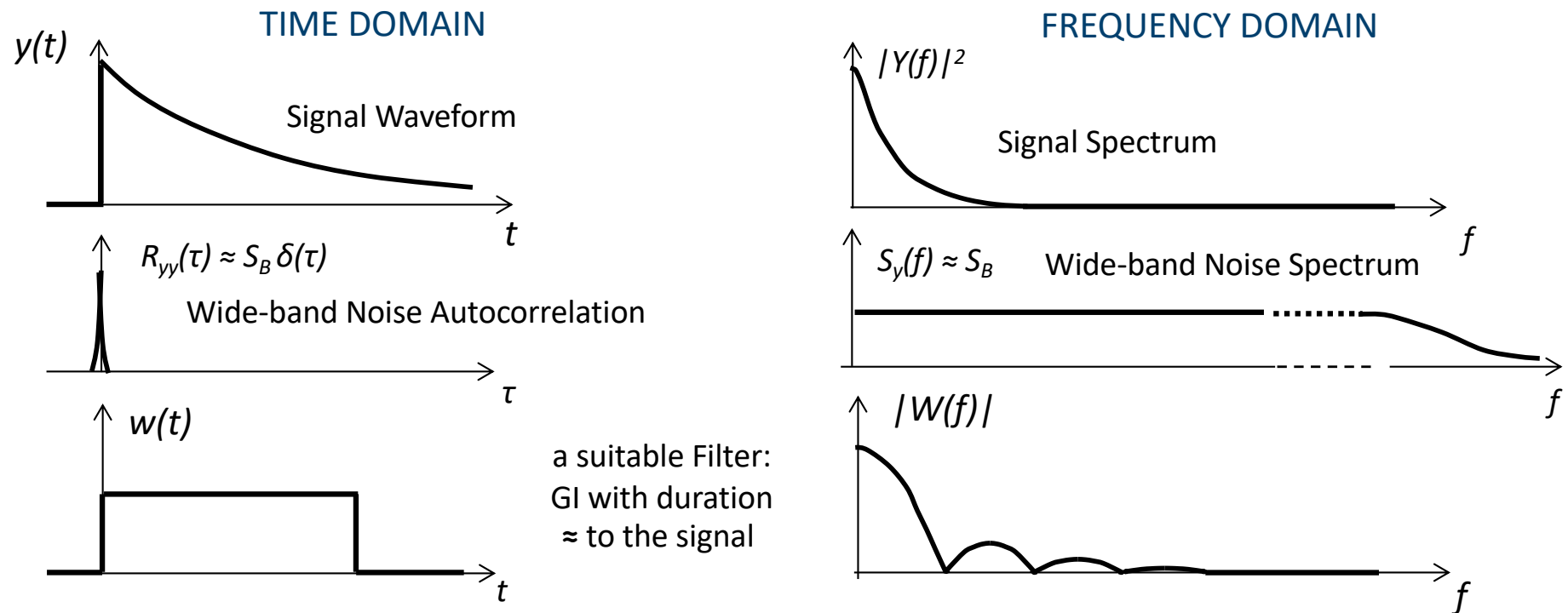
Filtering to measure Amplitude of Signals beset by White Stationary Noise

Let's consider first a basic case: **pulse signals** accompanied by **stationary white noise**. This ideal case is a good approximation for **real cases** where pulse signals are accompanied by wide-band noise, i.e. noise with

- Narrow autocorrelation, i.e. **width much smaller than the signal duration**
- Wideband uniform spectrum, i.e. **upper bandlimit much higher than that of the signal**



Filtering to measure Amplitude of Signals in White Noise



For a measurement of pulse amplitude, a filter has to collect most of the signal and reject most of the noise. It's intuitive that its action should be:

- as seen in time, more or less to average the signal and the white noise over the time interval occupied by the signal
- as seen in frequency, more or less to pass the low frequency range occupied by the signal and cut the higher frequency range where only white noise is present

This means that it's a **low-pass filter (LPF) tailored to the signal** (see GI example above)

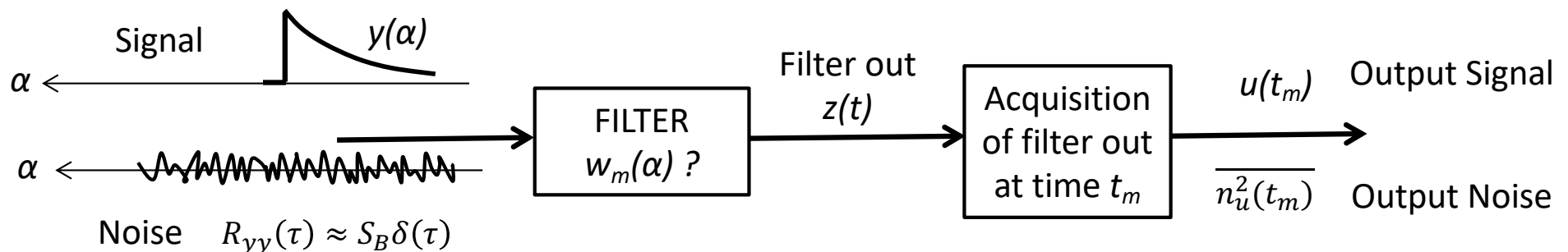
Low-pass filters tailored to the signal are suitable, but we'd like to know more, since basic questions are still open:

- *is there an optimal filter and if yes, what is it?*
- *If yes, what is the **best obtainable result**? That is, what is the optimized S/N and what is the smallest measurable amplitude?*

The issue is to find out the optimal weighting function, since it completely characterizes a linear filter.

Let's set in evidence the signal area A and the normalized waveform $b(t)$

$$y(t) = A \cdot b(t) \quad \text{with} \quad \int_{-\infty}^{\infty} b(t) dt = 1$$



QUESTION: is there a weighting function $w_m(\alpha)$ that optimizes $\frac{S}{N} = \frac{u(t_m)}{\sqrt{\overline{n_u^2(t_m)}}} ?$

Optimum Filtering with White Stationary Noise

The signal and noise acquired in the measurement are

$$u(t_m) = \int_{-\infty}^{\infty} y(\alpha) w_m(\alpha) d\alpha = A \cdot \int_{-\infty}^{\infty} b(\alpha) w_m(\alpha) d\alpha = A \cdot k_{bw}(0)$$

$$\overline{n_u^2(t_m)} = \int_{-\infty}^{\infty} R_{yy}(\alpha) k_{ww}(\alpha) d\alpha = S_B \cdot \int_{-\infty}^{\infty} w_m^2(\alpha) d\alpha = S_B \cdot k_{ww}(0)$$

Therefore

$$\left(\frac{S}{N}\right)^2 = \frac{u^2(t_m)}{\overline{n_u^2(t_m)}} = \frac{A^2}{S_B} \cdot \frac{k_{bw}^2(0)}{k_{ww}(0)}$$

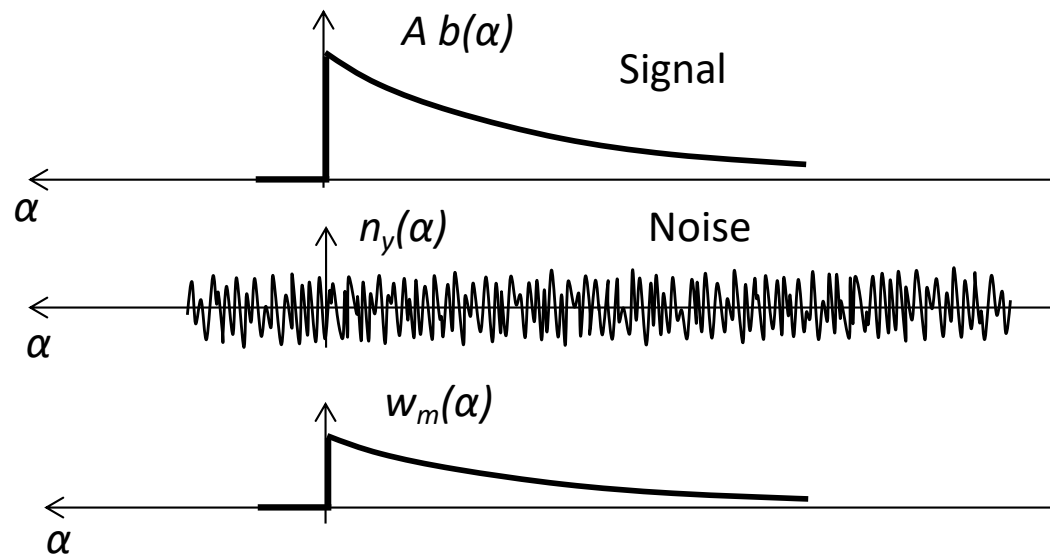
The $w_m(\alpha)$ that optimizes S/N for a given pulse shape $b(\alpha)$ is found by exploiting the known property of correlation functions (based on Schwartz's inequality)

$$k_{bw}^2(0) \leq k_{bb}(0) \cdot k_{ww}(0) \quad \text{that is} \quad \frac{k_{bw}^2(0)}{k_{ww}(0)} \leq k_{bb}(0)$$

where the maximum is achieved with **filter weighting proportional to the signal shape**

$$w_m(\alpha) \propto b(\alpha) \quad \text{which normalized to unit area is} \quad \boxed{w_m(\alpha) = b(\alpha)}$$

$$\text{and gives} \quad \max[k_{bw}^2(0)] = k_{bb}(0) \quad \text{that is} \quad \max \left[\frac{k_{bw}^2(0)}{k_{ww}(0)} \right] = k_{bb}(0)$$



The best result in measurements of the amplitude of signal pulses accompanied by **stationary** white noise is obtained with weighting function equal to the signal shape. This conclusion is intuitive: since the noise is uncorrelated, the output noise power is the weighted sum of the noise instantaneous power at all times; since this power is equal at all times, it is convenient to give higher weight when the signal is higher.

The filter with weighting function $w_m(\alpha)$ matched to the signal shape $b(\alpha)$

$$w_m(\alpha) \propto b(\alpha)$$

is indeed called **MATCHED FILTER**

The optimum S/N provided by the Matched Filter is

$$\left(\frac{S}{N}\right)_{opt}^2 = \frac{A^2}{S_B} \cdot k_{bb}(0) = \frac{A^2}{S_B} \cdot \int_{-\infty}^{\infty} b^2(\alpha) d\alpha$$



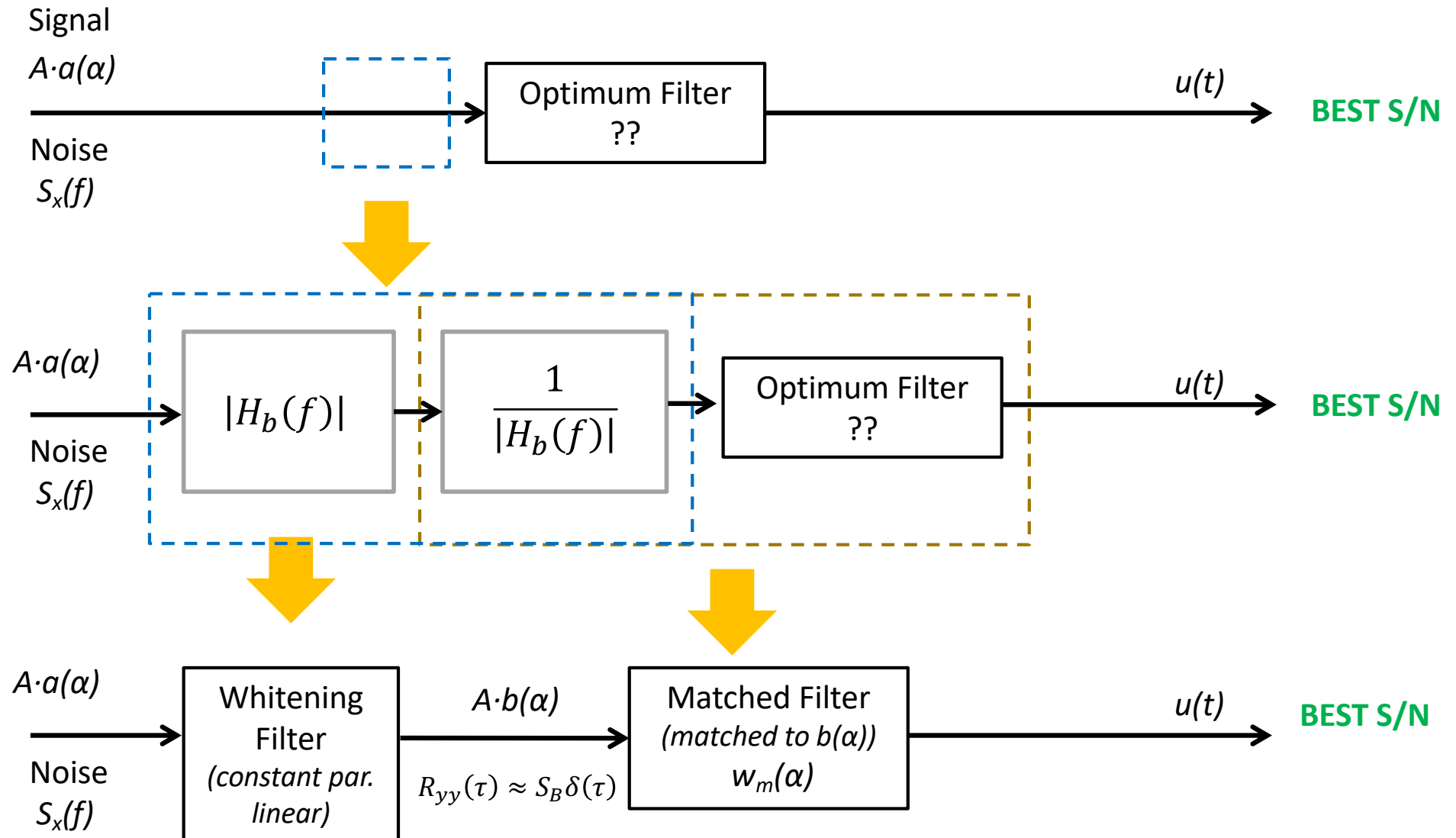
The minimum measurable amplitude A_{min} is defined as the amplitude that gives $(S/N)_{opt} = 1$, therefore it is

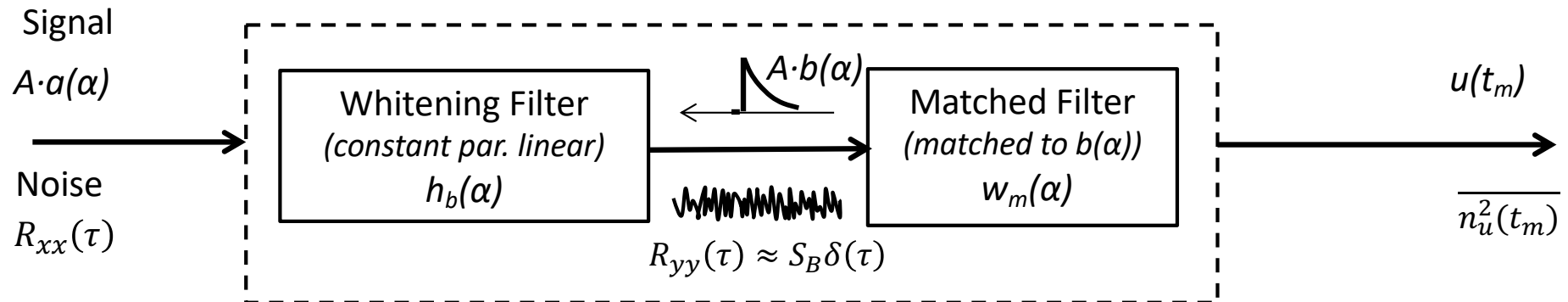
$$A_{min} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} b^2(\alpha) d\alpha}} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} B^2(f) df}}$$

Optimum Filtering with any Stationary Noise

Optimum Filtering with Any Stationary Noise

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- for a given stationary noise, it is always possible to find a **constant-parameter** linear filter that produces such a result, since it has transfer function $H_b(f)$ such that

$$|H_b(f)|^2 \propto \frac{1}{S_x(f)}$$

- the transformations performed by constant-parameter linear filters are reversible.
- Reversibility is essential:** nothing is lost in the transformation and whatever is done by the whitening filter can be reversed by the following filters.
- The whitening filter **modifies the waveform of the signal**, hence the following filter is **matched to this modified signal, not to the input signal!**
- The subdivision of the optimum filter in whitening filter and matched filter is a useful theoretical approach for analyzing the problem and finding the overall optimal weighting, but it is **NOT THE NECESSARY STRUCTURE** of the the optimum filter.

- **In conclusion**, the optimum weighting function for measuring the amplitude of pulses with any stationary noise is obtained as the **overall weighting function of two cascaded filter** stages: a **whitening** filter followed by a **matched** filter.
- For a given noise with spectrum $S_x(f)$ the **whitening** filter is a **constant-parameter linear** filter that has transfer function $H_b(f)$ such that $|H_b(f)|^2 \propto 1/S_x(f)$ (in time domain: filter autocorrelation function $k_{bb}(\tau)$ such that the convolution with the noise autocorrelation $R_{xx}(\tau)$ produces a δ -like autocorrelation $R_{yy}(\tau) \propto \delta(\tau)$)
- The action of the whitening filter is more evident in cases where the actual noise results from white noise filtered by some circuit. (as we will see in the next lesson..)
- In many cases it is quite difficult to design the noise-whitening filter and even more difficult to implement it, due to practical limitations of the real components (limited linear dynamic range; noise in the circuit elements; etc.).