COURSE OUTLINE

- Introduction
- Signals and Noise
- **Filtering: LPF3 Switched-Parameter Averaging Filters**
- Sensors and associated electronics

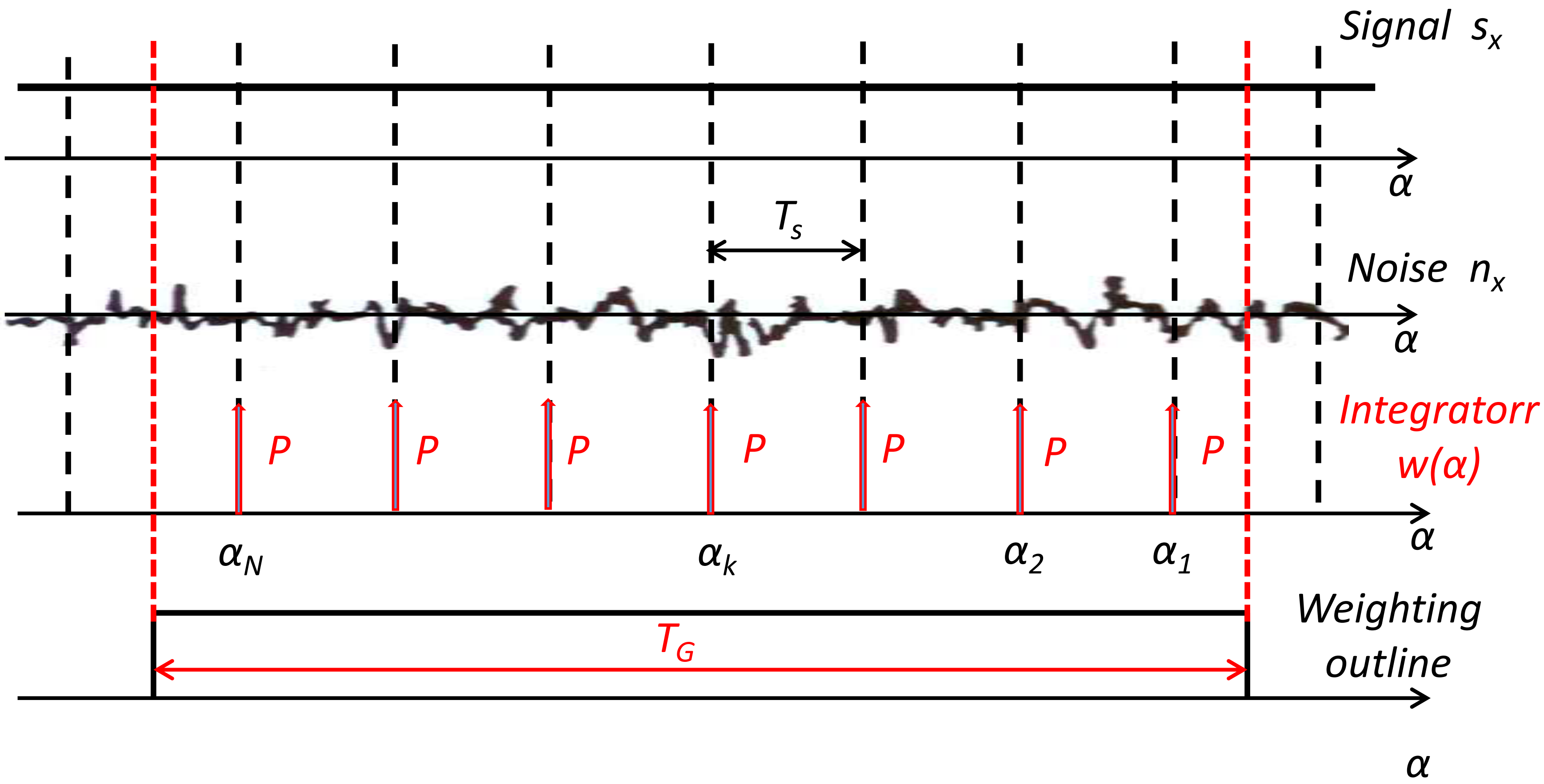
Switched-Parameter Averaging Filters

- Discrete Time Integrator (DTI)
- Discrete Time Integrator versus Gated Integrator
- Boxcar Integrator (BI)
- Ratemeter Integrator (RI)

Discrete-Time Integrator DI

Discrete-Time Integrator DI

It is the **discrete-time** equivalent of a continuous gated integrator with gate $T_G = N\, T_s$



- Samples taken with sampling frequency $f_s = 1/T_s$ i.e. at intervals T_s within T_G
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n << T_s$)
- Every sample is multiplied by P and summed, up to a total $\boldsymbol{N = T_G / T_s}$ **samples**

Discrete-Time Integrator

With **white noise**, the GI gives $S/N \propto \sqrt{T_G}$; we show now that the DI gives $S/N \propto \sqrt{N}$

The **output signal** is

$$\boldsymbol{s_y = N \cdot P s_x} \quad \text{(that is, the DC gain is } G = N \cdot P\text{)}$$

The **output noise** is $n_y = \sum_{k=1}^{N} P \cdot n_{xk}$ and

$$\overline{n_y{}^2} = P^2\overline{(n_{x1}{}^2 + n_{x2}{}^2 \ldots + n_{x1}n_{x2} + \cdots)} = P^2(\overline{n_{x1}{}^2} + \overline{n_{x2}{}^2} + \ldots + \overline{n_{x1}n_{x2}} + \cdots)$$

The noise samples are not correlated

$$\overline{n_{x1}n_{x2}} = \overline{n_{x2}n_{x3}} = \cdots = 0$$

and the noise is stationary $\overline{n_{x1}{}^2} = \overline{n_{x2}{}^2} = \cdots = \overline{n_x{}^2}$

Therefore

$$\boldsymbol{\overline{n_y{}^2} = N \cdot P^2 \overline{n_x{}^2}}$$

By summing N samples the signal is increased by N and the rms noise by $\sqrt{N}$

The SNR is thus improved by the factor $\sqrt{N}$

$$\boldsymbol{\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y{}^2}}} = \frac{N \cdot P s_x}{\sqrt{N \cdot P^2 \overline{n_x{}^2}}} = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x}$$

Discrete-Time Averager

An averager is simply a discrete-time integrator with sampling weight *P* adjusted to give unity DC gain, that is $G = N \cdot P = 1$

$$P = \frac{1}{N}$$

and therefore output signal equal to input

$$\boldsymbol{s_y} = \boldsymbol{s_x}$$

The output noise is reduced to

$$\overline{n_y{}^2} = N \cdot P^2 \overline{n_x{}^2} = \frac{1}{N} \cdot \overline{n_x{}^2}$$

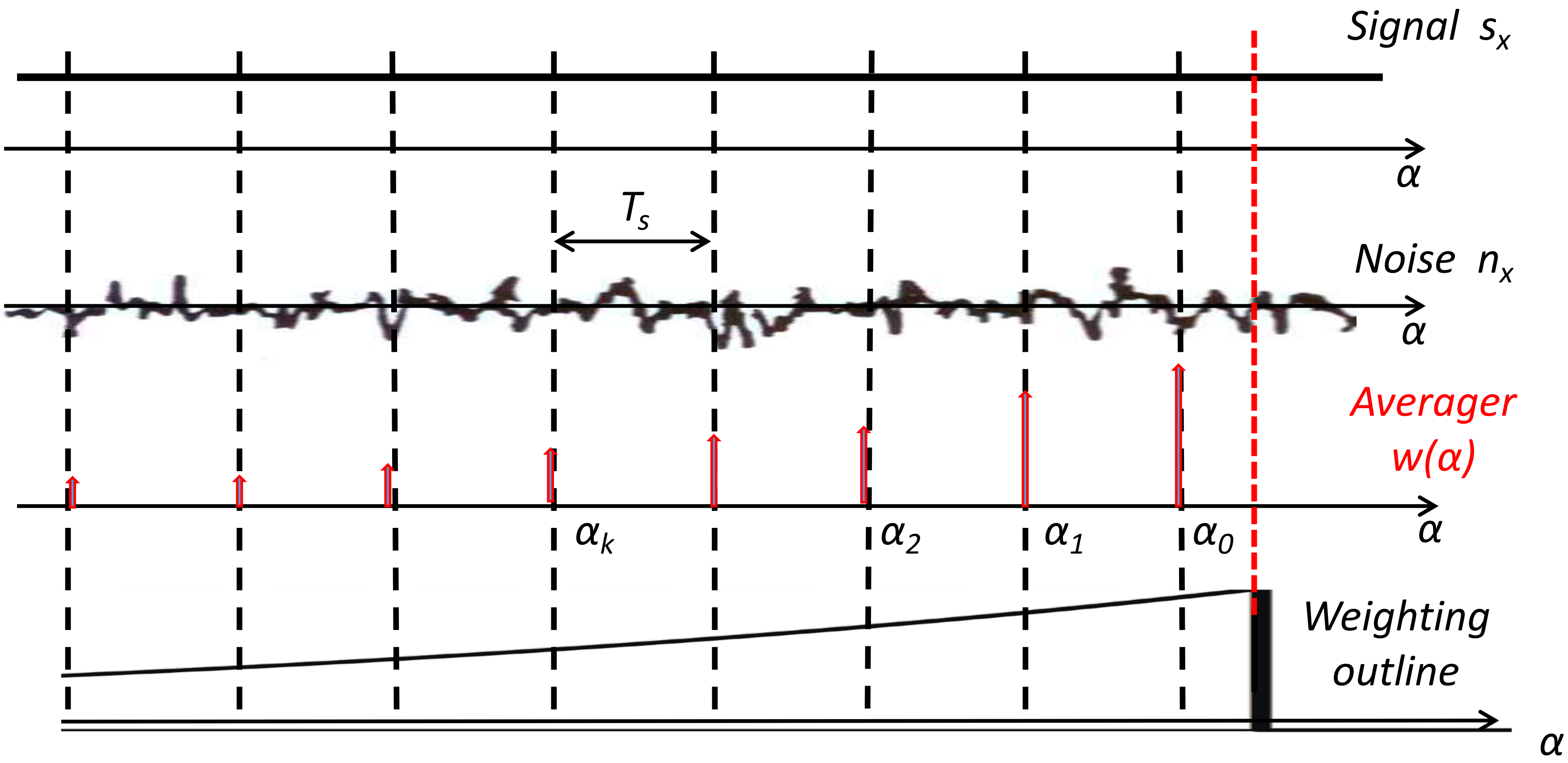
$$\sqrt{\overline{\boldsymbol{n_y}^{\boldsymbol{2}}}} = \frac{\sqrt{\overline{\boldsymbol{n_x}^{\boldsymbol{2}}}}}{\sqrt{\boldsymbol{N}}}$$

which corresponds to the enhancement of the S/N

$$\left(\frac{S}{N}\right)_y = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

Discrete-Time Exponential Averager

It is the discrete-time equivalent of an RC integrator



- Samples are taken with sampling frequency $f_s=1/T_s$ i.e. at intervals T_s
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n$<< T_s)
- The sample **weight slowly decays** with the sample «age»: $w_k = Pr^k$ **with $(1-r) << 1$**

Discrete-Time Exponential Averager

Output signal $s_y = s_x \cdot P \cdot \sum_{k=0}^{\infty} r^k = s_x \cdot P \frac{1}{1-r}$ (i.e, DC gain $G = P \frac{1}{1-r}$)

Output mean square noise

$$\overline{n_y{}^2} = P^2 (\overline{n_{x0}{}^2} + r^2 \cdot \overline{n_{x1}{}^2} + \cdots + r^{2k} \cdot \overline{n_{xk}{}^2} + \cdots + r^k r^j \cdot \overline{n_{xk} n_{xj}} + \cdots)$$

The noise samples are not correlated ($\overline{n_{xk} n_{xj}} = 0$ for $k \neq j$)

and the noise is stationary ($\overline{n_{x0}{}^2} = \overline{n_{x1}{}^2} = \cdots = \overline{n_x{}^2}$)

Therefore

$$\overline{n_y{}^2} = \overline{n_x{}^2} \cdot P^2 \left(1 + r^2 + \ \ldots + r^{2k} + \ \ldots\right) = \overline{n_x{}^2} \cdot P^2 \cdot \frac{1}{1-r^2}$$

The SNR is thus improved to

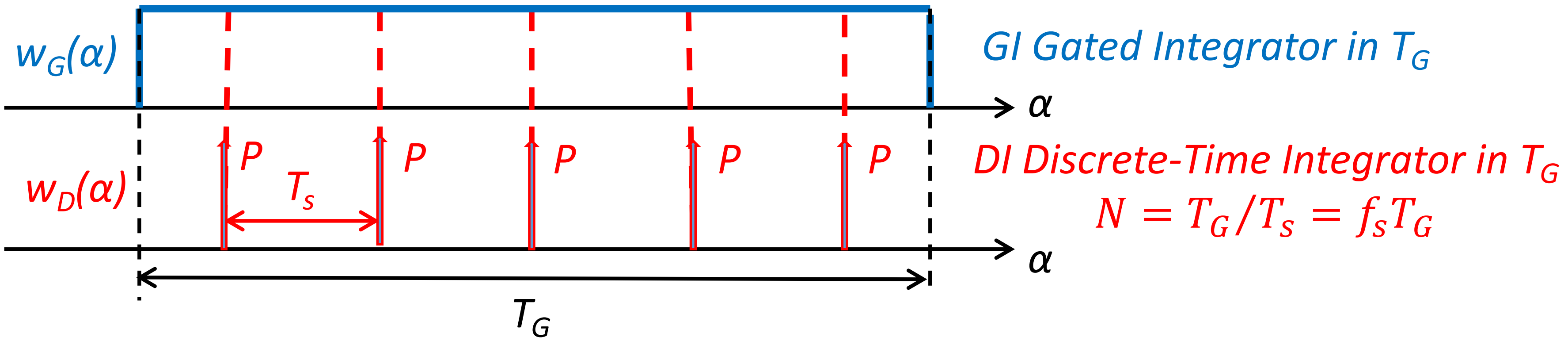
$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y^2}}} = \frac{P s_x}{1-r} \frac{1}{\sqrt{\frac{\overline{n_x^2} P^2}{1-r^2}}} = \left(\frac{S}{N}\right)_x \sqrt{\frac{1+r}{1-r}}$$

But the attenuation ratio r is very close to unity ($1 - r) << 1$ hence ($1 + r) \approx 2$ and therefore

$$\boldsymbol{\left(\frac{S}{N}\right)_y \cong \left(\frac{S}{N}\right)_x \sqrt{\frac{2}{1-r}}}$$

Discrete-Time Integrator versus GI

Discrete-Time Integrator vs. GI



INPUT: DC signal s_x and wide-band noise S_b (bandwidth $2f_n >> f_s$, correlation width $2T_n << T_s$) with rms value $\overline{n_x{}^2} = S_b 2f_n = S_b/2T_n$

With unity DC gain $s_y = s_x$

- Noise reduction by GI $\sqrt{\overline{n_{yG}{}^2}} = \sqrt{\overline{n_x{}^2}} \Big/ \sqrt{\frac{T_G}{2T_n}}$
- Noise reduction by DI $\sqrt{\overline{n_{yD}{}^2}} = \sqrt{\overline{n_x{}^2}} \Big/ \sqrt{N}$

Discrete-Time Integrator vs. GI

The improvement factor is

- $\sqrt{N}$ for the DI, increasing with the number N of samples taken
- $\sqrt{T_G/2T_n}$ for the GI, constant for a given T_G

QUESTION : is it possible to attain with a DI better S/N improvement than a GI just by increasing the number N (i.e. by using very fast sampling electronics)?

ANSWER: NO !!

In fact , since $N = T_G/T_s$ for having $N > T_G/2T_n$ it must be

$$T_s < 2T_n$$

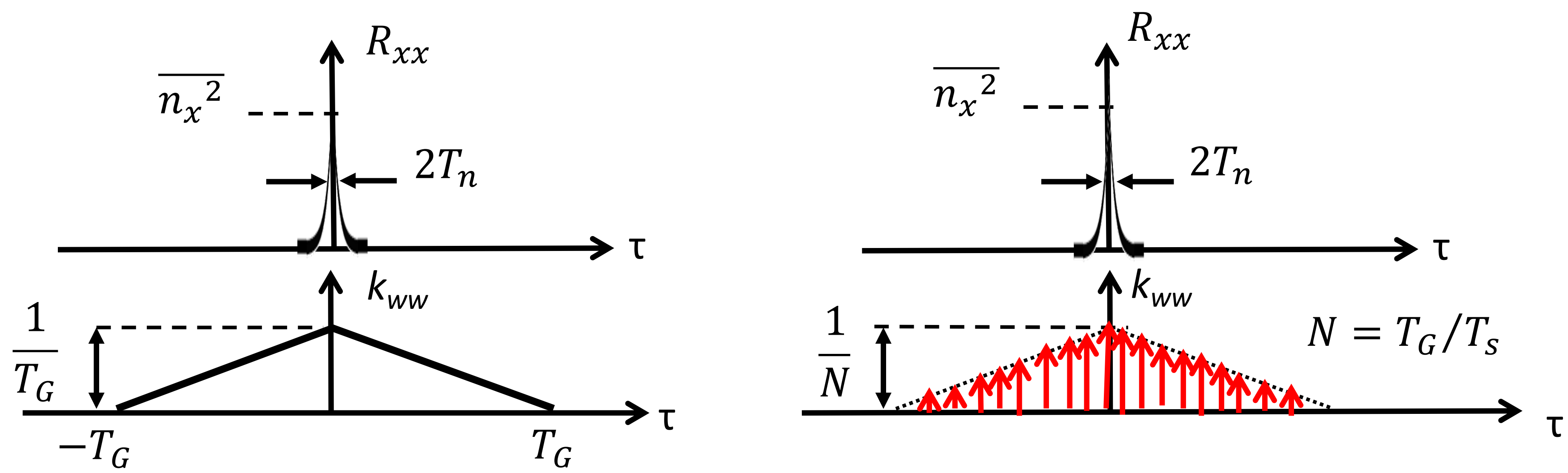
in these conditions

- the samples are no more uncorrelated
- the improvement factor is **no more given by $\sqrt{N}$**
- There is still an improvement factor, but it must be evaluated taking into account the correlation between the noise samples.
- It is anyway $(S/N)_{DI} \leq (S/N)_{GI}$ with $(S/N)_{DI} \rightarrow (S/N)_{GI}$ as N is increased, as we can demonstrate in time domain and in frequency domain

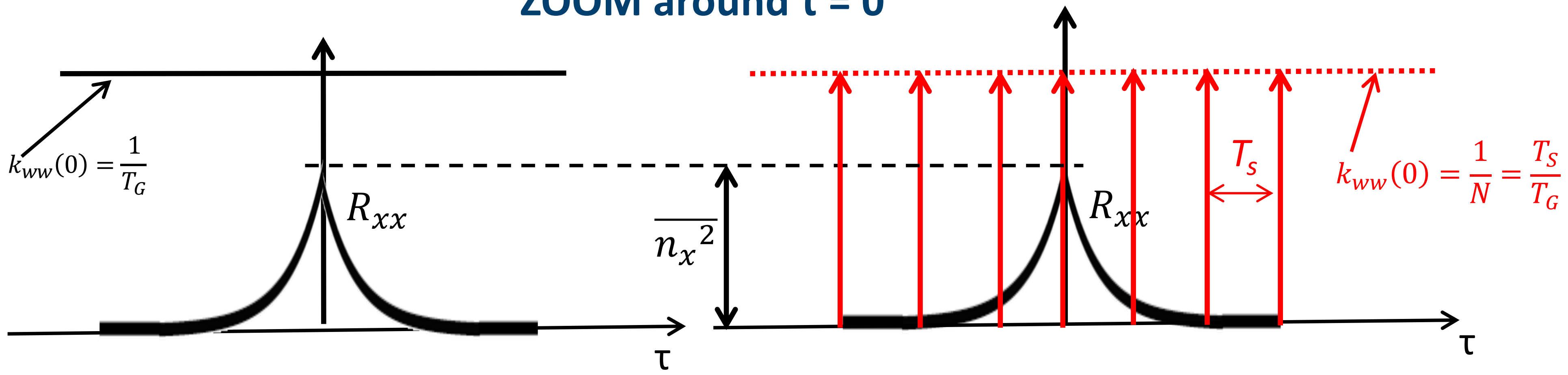
Discrete-Time Integrator vs. GI (time domain)

GI Gated Integrator
(normalized to unity DC gain G=1)

DI Discrete-time Integrator
(normalized to unity DC gain G=1)

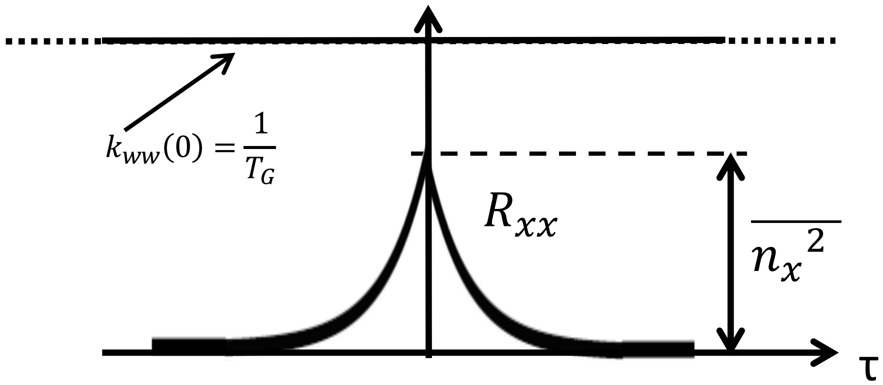


ZOOM around τ = 0



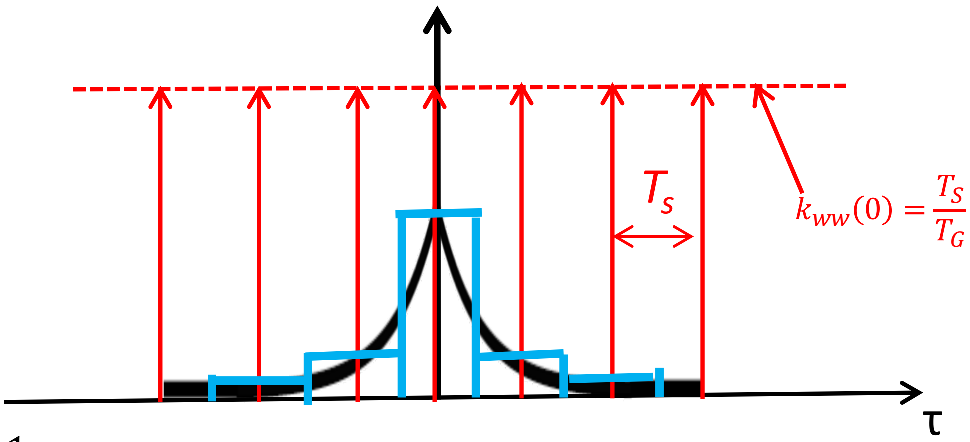
Discrete-Time Integrator vs. GI (time domain)

GI Gated Integrator (with G=1)



$$\overline{n_{yG}^2} = \frac{1}{T_G} \cdot \ (area\ of\ R_{xx})$$

DI Discrete-time Integrator (with G=1)



$$\overline{n_{yD}^2} = \frac{1}{T_G} \cdot \ T_S \cdot (sum\ of\ R_{xx}\ samples\ at\ \tau = 0;\ \pm T_S;\ \pm 2T_S; \ldots.)$$

$$\overline{n_{yD}^2} = \frac{1}{T_G} \cdot \ (area\ of the\ scaloid that approximates R_{xx})$$

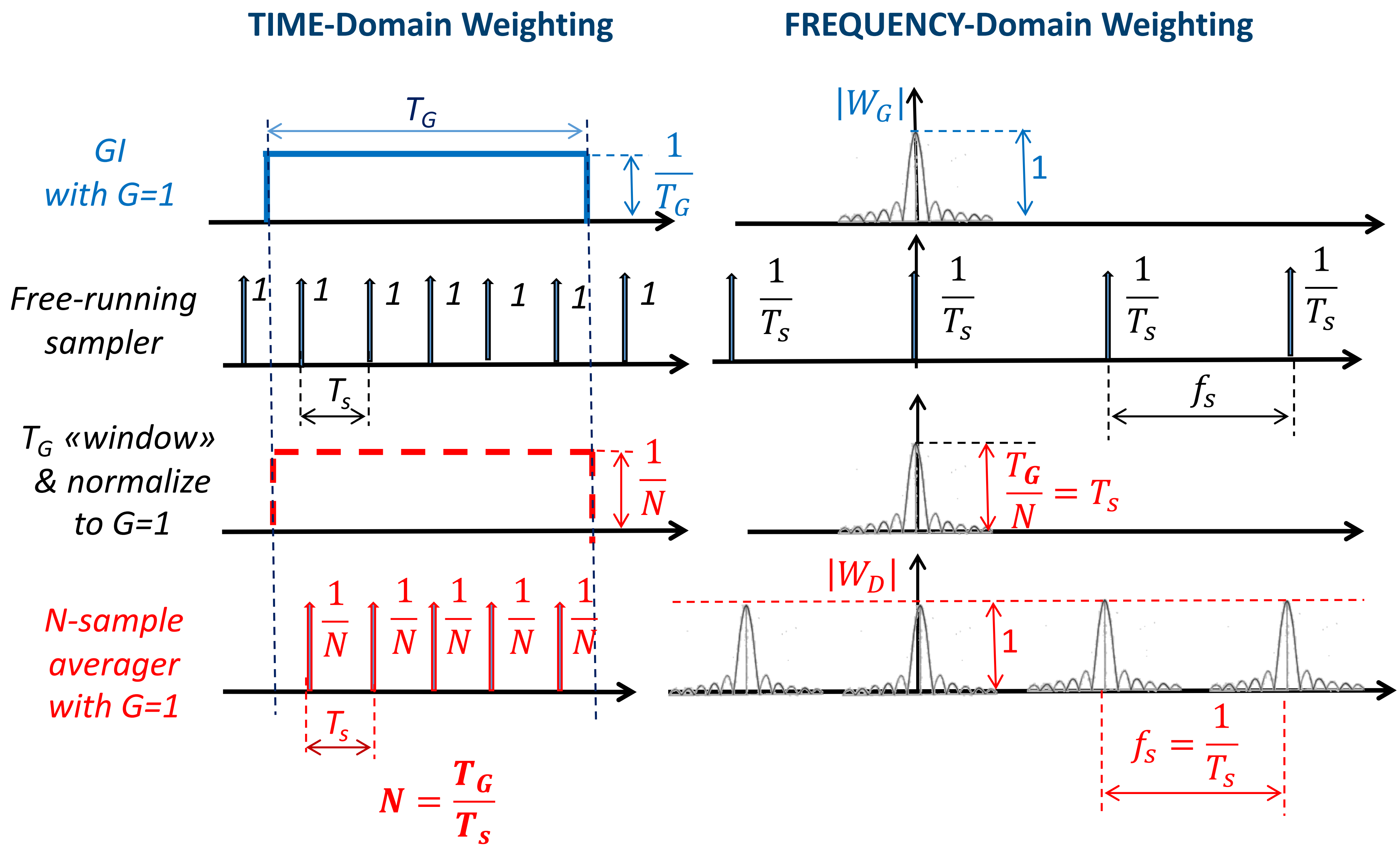
The scaloid area is greater than the R_{xx} area, therefore

$$\overline{n_{yD}^2} \geq \overline{n_{yG}^2} = \overline{n_x^2} \cdot \frac{2T_n}{T_G}$$

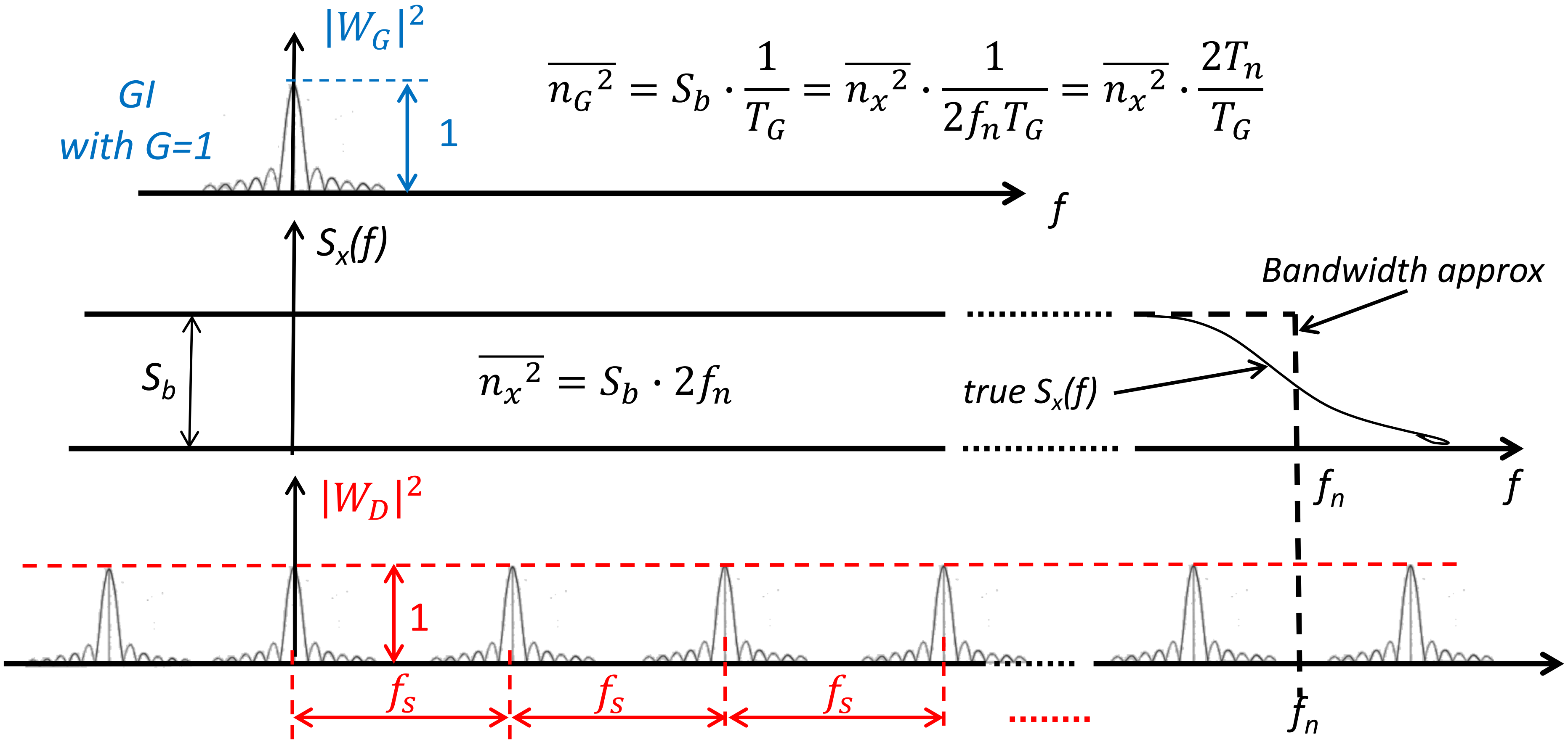
with

$$\overline{n_{yD}^2} \rightarrow \overline{n_{yG}^2} \quad as \quad T_S \rightarrow 0$$

Discrete-Time Integrator vs. GI (frequency domain)



Noise filtering analysis: GI vs. DI (frequency domain)



$$\overline{n_G{}^2} = S_b \cdot \frac{1}{T_G} = \overline{n_x{}^2} \cdot \frac{1}{2 f_n T_G} = \overline{n_x{}^2} \cdot \frac{2T_n}{T_G}$$

$$\overline{n_x{}^2} = S_b \cdot 2 f_n$$

$$\overline{n_G{}^2} = \int_{-\infty}^{\infty} S_x(f)\,|W_G(f)|^2 df \qquad \overline{n_D{}^2} = \int_{-\infty}^{\infty} S_x(f)\,|W_D(f)|^2 df$$

The figure illustrates how the output noise $\overline{n_D{}^2}$ is reduced and S/N is enhanced by increasing the sampling frequency f_s (for a given averaging time T_G)

Noise filtering analysis: GI vs. DI

a) As long as $f_s \ll f_n$:

- the noise samples are uncorrelated
- each line of $|W_D|^2$ is identical to $|W_G|^2$ of the GI (with same DC gain G=1)
- a high number N_L of lines of $|W_D|^2$ falls within the noise bandwidth $2f_n$
- the output noise of the DI is N_L times that of the GI

$$\overline{n_D{}^2} = \overline{n_G{}^2} \cdot N_L$$

With good approximation it is

$$N_L \cong 2f_n/f_s$$

and it is confirmed that for uncorrelated samples the S/N increases as $\sqrt{N}$

$$\overline{n_D{}^2} = \overline{n_x{}^2} \cdot \frac{1}{T_G f_s} = \frac{\overline{n_x{}^2}}{N}$$

b) When f_s becomes comparable to f_n or higher

- the previous result is no more valid.
- the output noise must be computed with the actual noise spectrum

$$\overline{n_D{}^2} = \int_{-\infty}^{\infty} S_x(f)\,|W_D(f)|^2 df \geq \overline{n_G{}^2}$$

- The figure shows that $\overline{n_D{}^2}$ is always higher than $\overline{n_G{}^2}$ and attains it for $f_s \to \infty$

$$\lim_{f_s \to \infty} \overline{n_D{}^2} = \overline{n_G{}^2}$$

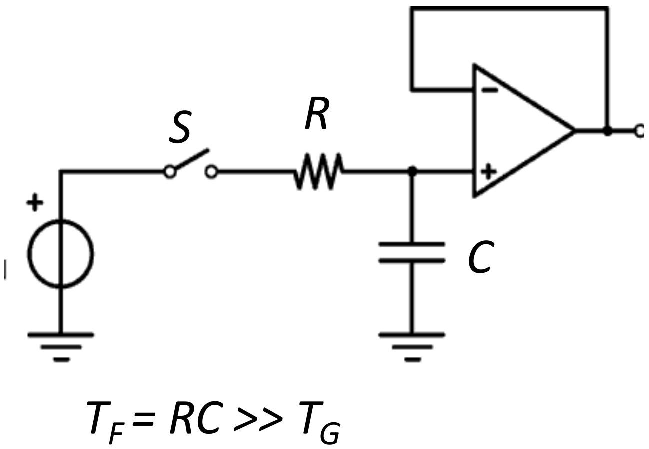
Boxcar Integrator BI

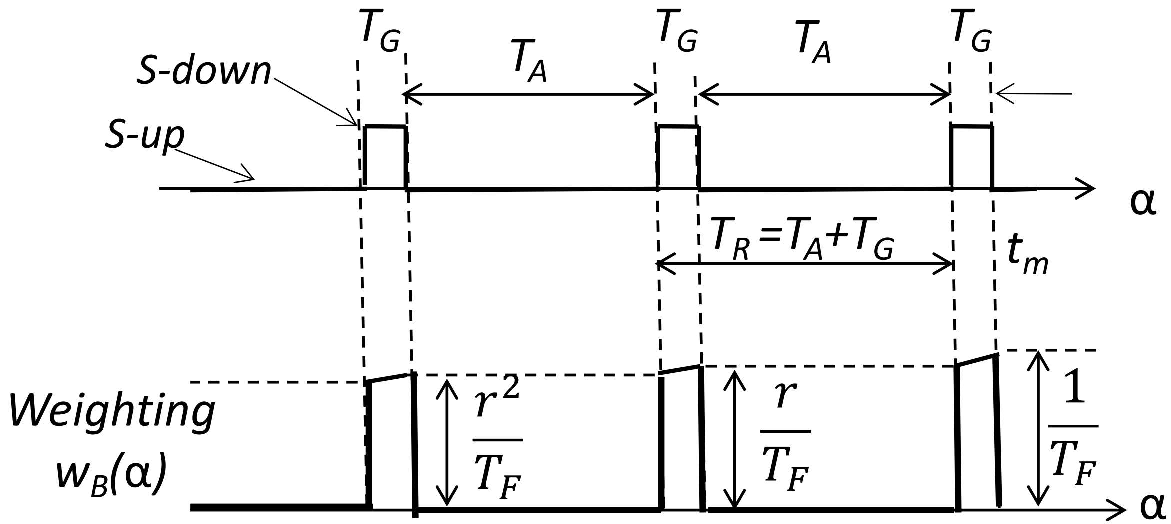
Boxcar Integrator (BI)

This simple analog circuit combines two functions:

1. Sample Acquisition by gated integration
2. Exponential **averaging of samples**

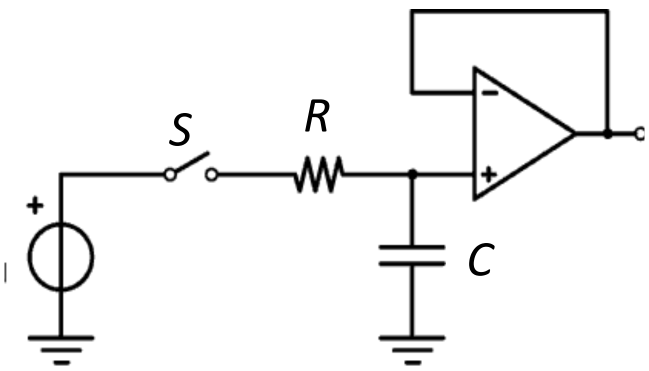
The circuit employed is the same of the Gated Integrator, but with a fundamental difference: the capacitor is **NOT RESET between the acquisitions.**

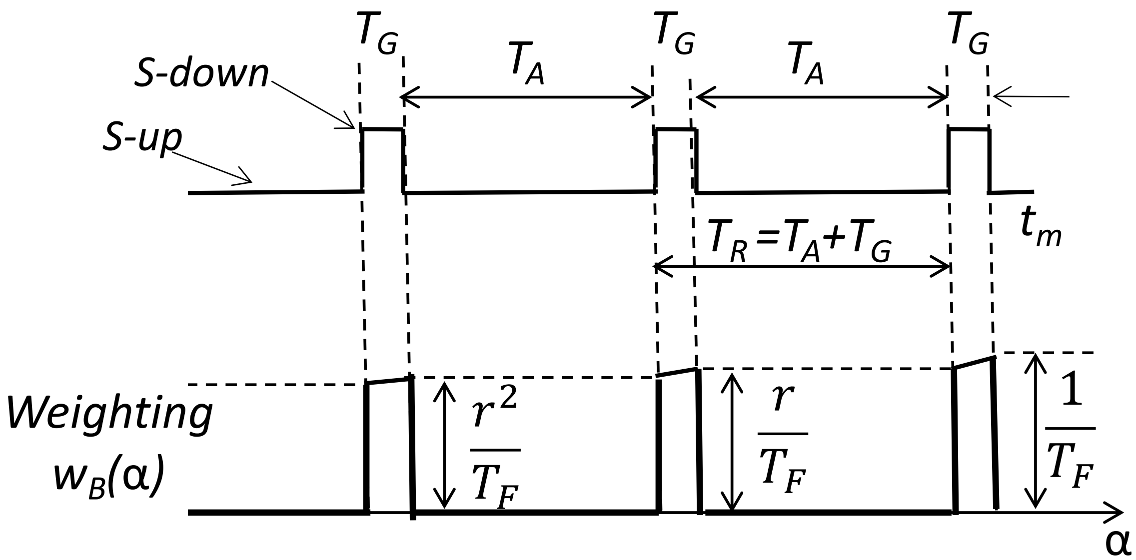




- In T_A the C is in HOLD state: nothing changes, no memory loss and no new charge input
- In T_G the discharge of C (memory loss) reduces the previously stored value by the factor $r = e^{-T_G/T_F}$. **NB**: r does NOT depend on the interval T_A

Boxcar Integrator (BI)





- BI behaves as RC-integrator (RCI) when the switch is closed (S-down); it is in HOLD state when the switch is open (S-up)
- In fact, the weighting function $w_B(\alpha)$ of the BI is obtained by subdividing $w_{RC}(\alpha)$ of the RCI it in «slices» of width T_G and placing them over the S-down intervals
- G=1 : the DC gain of BI (area of w_B) is unity (like that of RCI): the BI is an averager
- The autocorrelation functions k_{wwB} of BI and k_{wwRC} of RCI are very different, but have equal central value $k_{ww}(0)$

$$k_{wwB}(0) = k_{wwRC}(0) = \frac{1}{2RC} = \frac{1}{2T_F}$$

Boxcar Integrator (BI): S/N enhancement

The input wide-band noise S_b with bandwidth $2f_n$, autocorrelation width $2T_n$, has mean square value

$$\overline{n_x{}^2} = S_b \cdot \frac{1}{2T_n}$$

The BI output noise is

$$\overline{n_y{}^2} = S_b \cdot k_{wwB}(0) = S_b \cdot 1/2T_F = \overline{n_x{}^2} \cdot \frac{T_n}{T_F}$$

Therefore, since BI has G=1 the S/N enhancement is

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

The S/N enhancement does **NOT** depend **on the RATE** of the samples because it is obtained by **averaging over a given number** of samples and **not over a given time interval**. In fact, counting the samples (from the measurement time t_m and going backwards) the sample weight is reduced below 1/100 for sample number > $4.6T_F/T_G$, irrespective of the sample rate

Boxcar Integrator (BI): S/N enhancement

The BI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the BI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_G/T_F} \cong 1 - T_G/T_F$$

which enhances the S/N by the factor

$$\sqrt{(1+r)/(1-r)} \cong \sqrt{2/(1-r)} = \sqrt{2T_F/T_G}$$

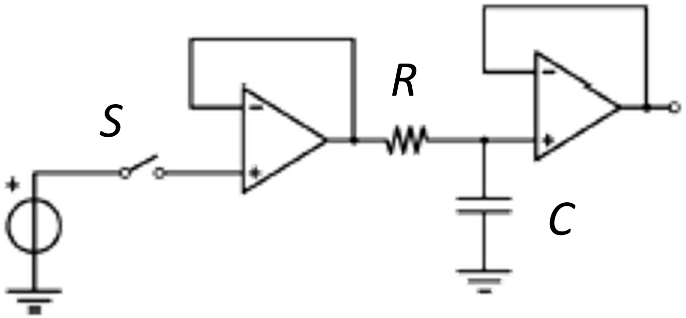
NB: this factor is INDEPENDENT of the RATE of samples, because the AVERAGE IS DONE ON A GIVEN NUMBER OF SAMPLES and not on a given time.

The S/N enhancement is thus confirmed and clarified

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_F}{T_G}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

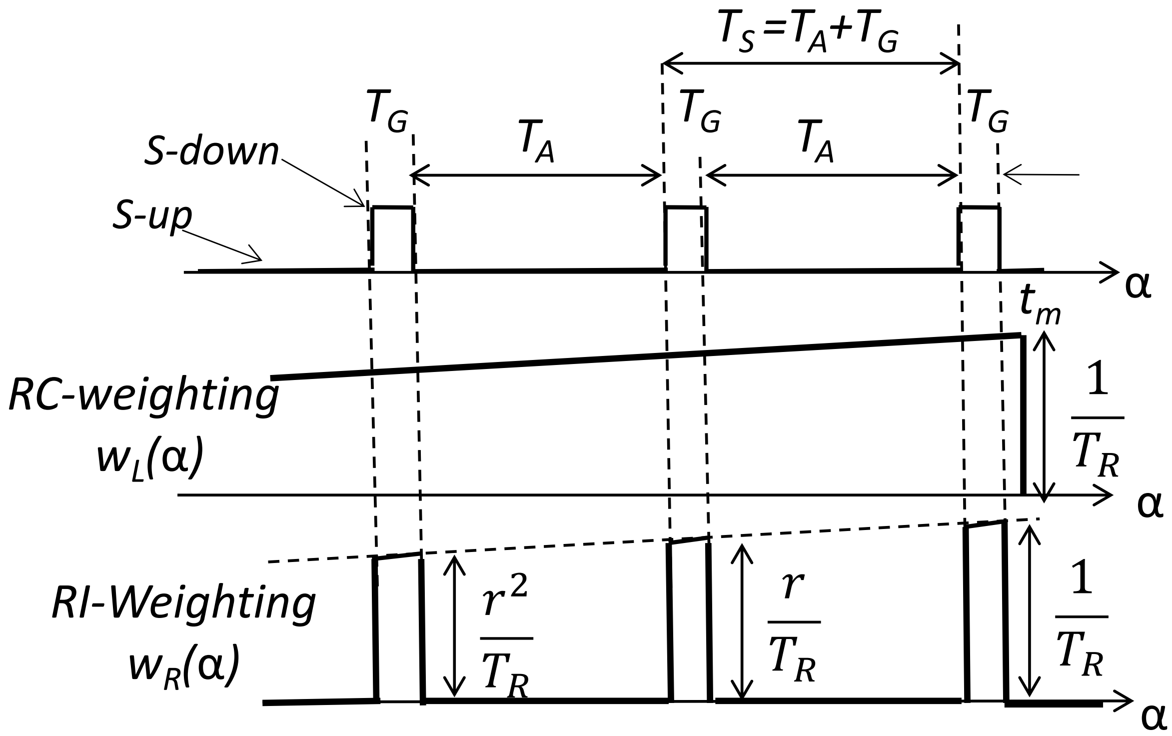
Ratemeter Integrator RI

Ratemeter Integrator (RI)



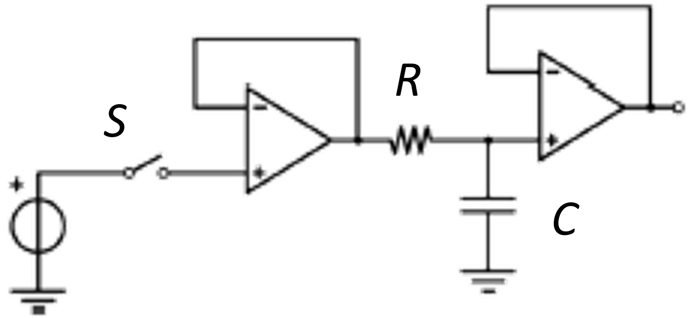
T_R = RC >> T_S

for averaging many samples



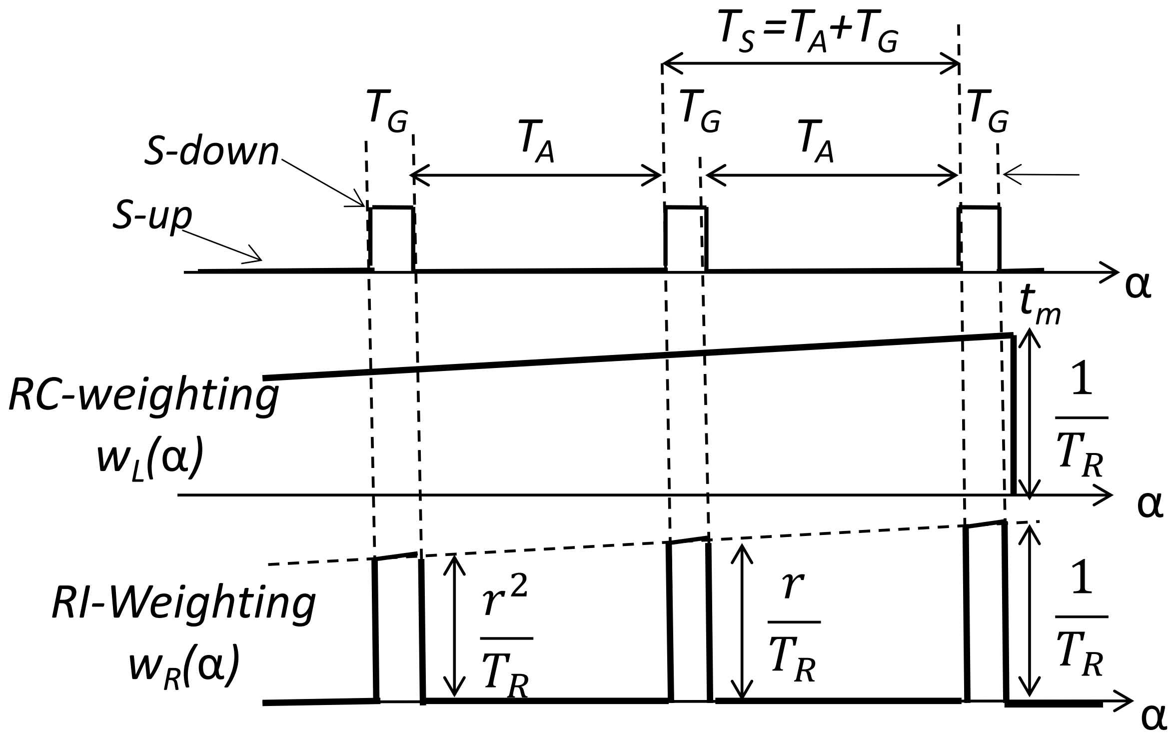
- By inserting a buffer between S and RC a new **exponential averager** is obtained, radically **different from BI**. The integrator is no more a switched-parameter RC filter: it is now a constant-parameter RC filter, unaffected by the switch S.
- There is no HOLD state. The memory loss goes on all the time; the weight reduction from sample to sample is $\boldsymbol{r = e^{-(T_G+T_A)/T_F} = e^{-T_S/T_F}}$. **NB:** ***r*** **DEPENDS** on the sample **RATE**!
- During T_G (with S-down) the input is integrated in C
  During T_A (with S-up) the input is NOT allowed

Ratemeter Integrator (RI)



$T_R = RC >> T_S$

for averaging many samples



- The **DC gain is G < 1** (the RC filter has G=1, but it receives just a fraction of the input!)
- With $T_R >> T_S$ the DC gain G is proportional to the sample rate $f_S = 1/T_S$

$$G = \int_{-\infty}^{\infty} w_R(\alpha)d\alpha \cong \frac{T_G}{T_S} \cdot \int_{-\infty}^{\infty} w_L(\alpha)d\alpha \cong \frac{T_G}{T_S} = f_S \cdot T_G$$

NB: if the input signal amplitude x_S is constant but f_S varies, the output signal y_S varies. In fact, the circuit is also employed as **analog ratemeter**: with constant input voltage x_S it produces a quasi DC output signal proportional to the repetition rate f_S

Ratemeter Integrator (RI): S/N enhancement

The RI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the RI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_S/T_R} \cong 1 - T_S/T_R$$

which enhances the S/N by the factor

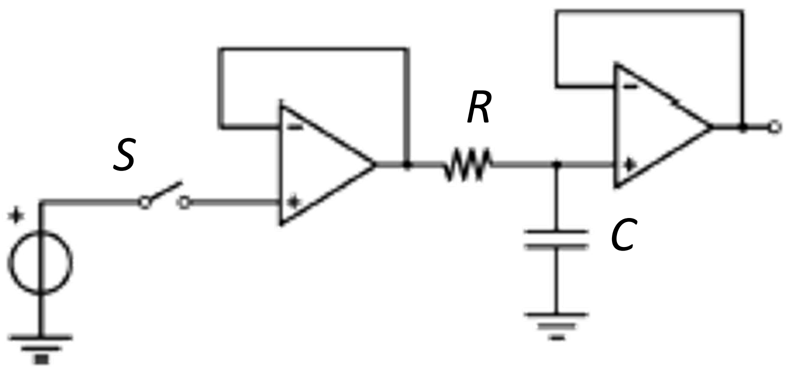
$$\sqrt{\frac{1+r}{1-r}} \cong \sqrt{\frac{2}{1-r}} = \sqrt{\frac{2T_R}{T_S}} = \sqrt{2T_R f_S}$$

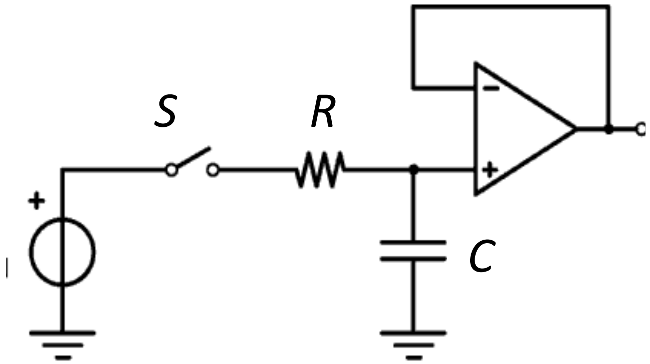
NB: this factor DEPENDS on the sample RATE f_S because the AVERAGE IS DONE ON A GIVEN TIME and not on a given number of samples. The weight reduction is below 1/100 for samples that at the measurement time t_m are «older» than 4.6·T_R

The S/N enhancement thus depends on the sample rate f_s

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_R}{T_S}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{f_S T_G \frac{T_R}{T_n}}$$

BI and RI: Passive Circuit comparison





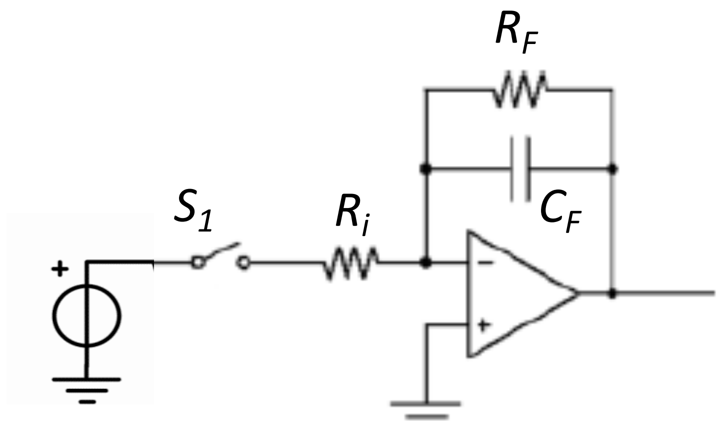
RATEMETER INTEGRATOR

- Switch S acts as gate on the input source
- Switch S is decoupled from the RC passive filter by the voltage buffer
- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the RC value

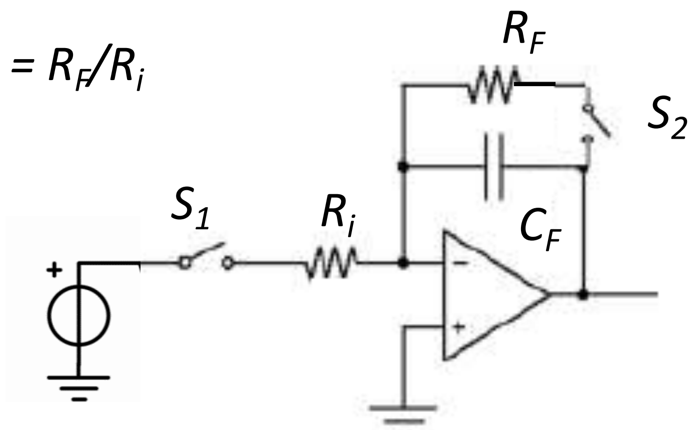
BOXCAR INTEGRATOR

- Switch S acts as gate on the input source
- Switch S acts also on the RC passive filter (changes the resistor value)
- The time constant T_F of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value

BI and RI: Active Circuit comparison



DC gain $G = R_F/R_i$



RATEMETER INTEGRATOR

- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- The R_FC_F integrator is unaffected by S_1; it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the R_FC_F value

BOXCAR INTEGRATOR

- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- A second switch S_2 is required for switching the time constant T_F of the integrator from finite R_FC_F (S_2-down) to infinite (S_2-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value