COURSE OUTLINE

- Introduction

- Signals and Noise

- **Filtering: LPF3 Switched-Parameter Averaging Filters**

- Sensors and associated electronics

Switched-Parameter Averaging Filters

- Discrete Time Integrator (DTI)

- Discrete Time Integrator versus Gated Integrator

- Boxcar Integrator (BI)

- Ratemeter Integrator (RI)

Discrete-Time Integrator DI

Discrete-Time Integrator DI

It is the **discrete-time** equivalent of a continuous gated integrator with gate $T_G = N\,T_s$
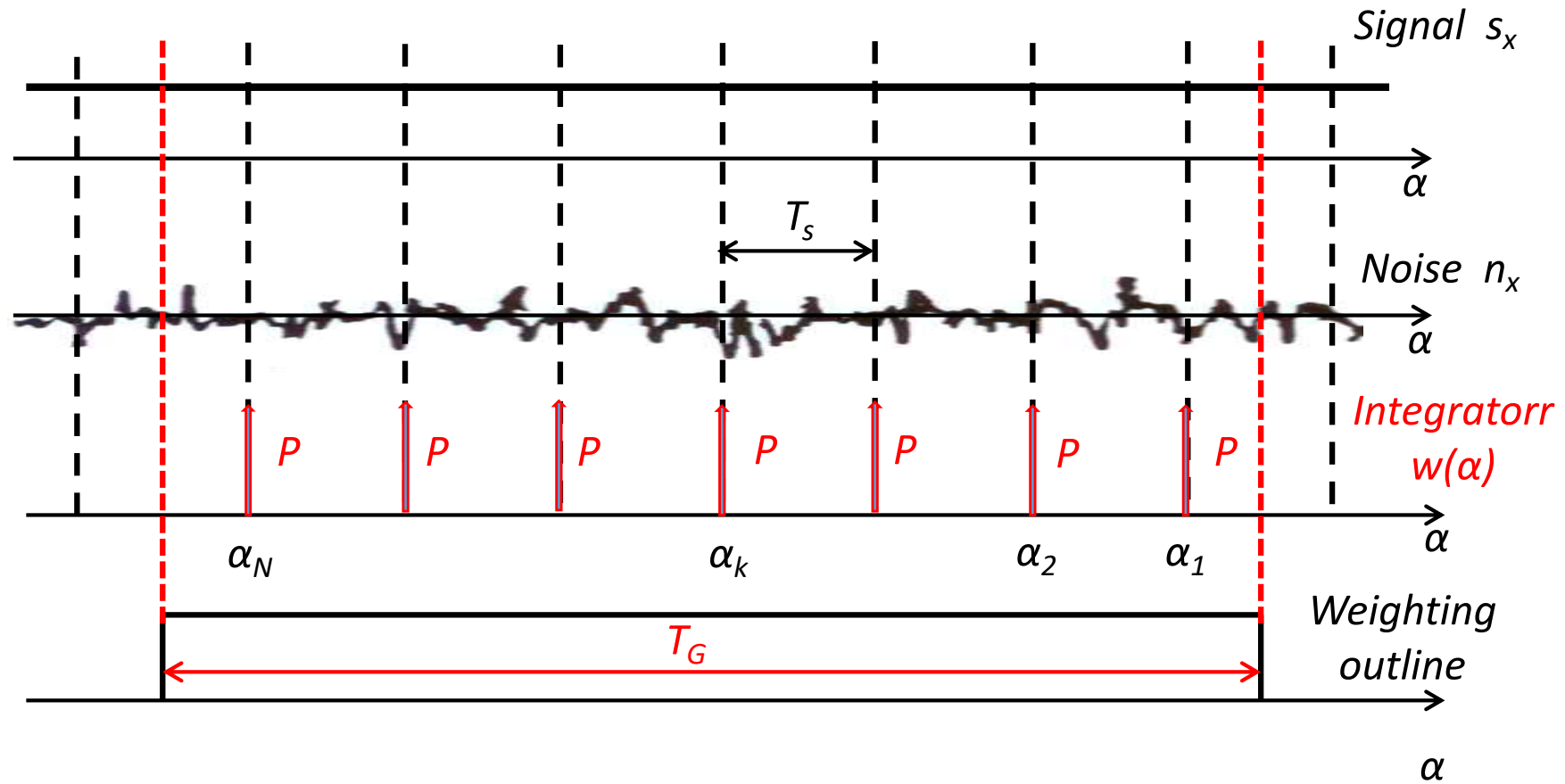
- Samples taken with sampling frequency $f_s = 1/T_s$ i.e. at intervals T_s within T_G
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n \ll T_s$)
- Every sample is multiplied by P and summed, up to a total **$N = T_G\,/\,T_s$ samples**

Discrete-Time Integrator

With **white noise**, the GI gives $S/N \propto \sqrt{T_G}$; we show now that the DI gives $S/N \propto \sqrt{N}$

The **output signal** is

$$\boldsymbol{s_y = N \cdot Ps_x} \quad \text{(that is, the DC gain is } G = N \cdot P\text{)}$$

The **output noise** is $\quad n_y = \sum_{k=1}^{N} P \cdot n_{xk} \quad$ and

$$\overline{n_y{}^2} = P^2 \overline{(n_{x1}{}^2 + n_{x2}{}^2 \ldots + n_{x1}n_{x2} + \cdots)} = P^2 (\overline{n_{x1}{}^2} + \overline{n_{x2}{}^2} + \ldots + \overline{n_{x1}n_{x2}} + \cdots)$$

The noise samples are not correlated

$$\overline{n_{x1}n_{x2}} = \overline{n_{x2}n_{x3}} = \cdots = 0$$

and the noise is stationary $\quad \overline{n_{x1}{}^2} = \overline{n_{x2}{}^2} = \cdots = \overline{n_x{}^2}$

Therefore

$$\boldsymbol{\overline{n_y{}^2} = N \cdot P^2 \overline{n_x{}^2}}$$

By summing N samples the signal is increased by N and the rms noise by $\sqrt{N}$
The SNR is thus improved by the factor $\sqrt{N}$

$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y{}^2}}} = \frac{N \cdot Ps_x}{\sqrt{N \cdot P^2 \overline{n_x{}^2}}} = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

REMEMBER

An averager is simply a discrete-time integrator with sampling weight P adjusted to give unity DC gain, that is $G = N \cdot P = 1$

$$P = \frac{1}{N}$$

and therefore output signal equal to input

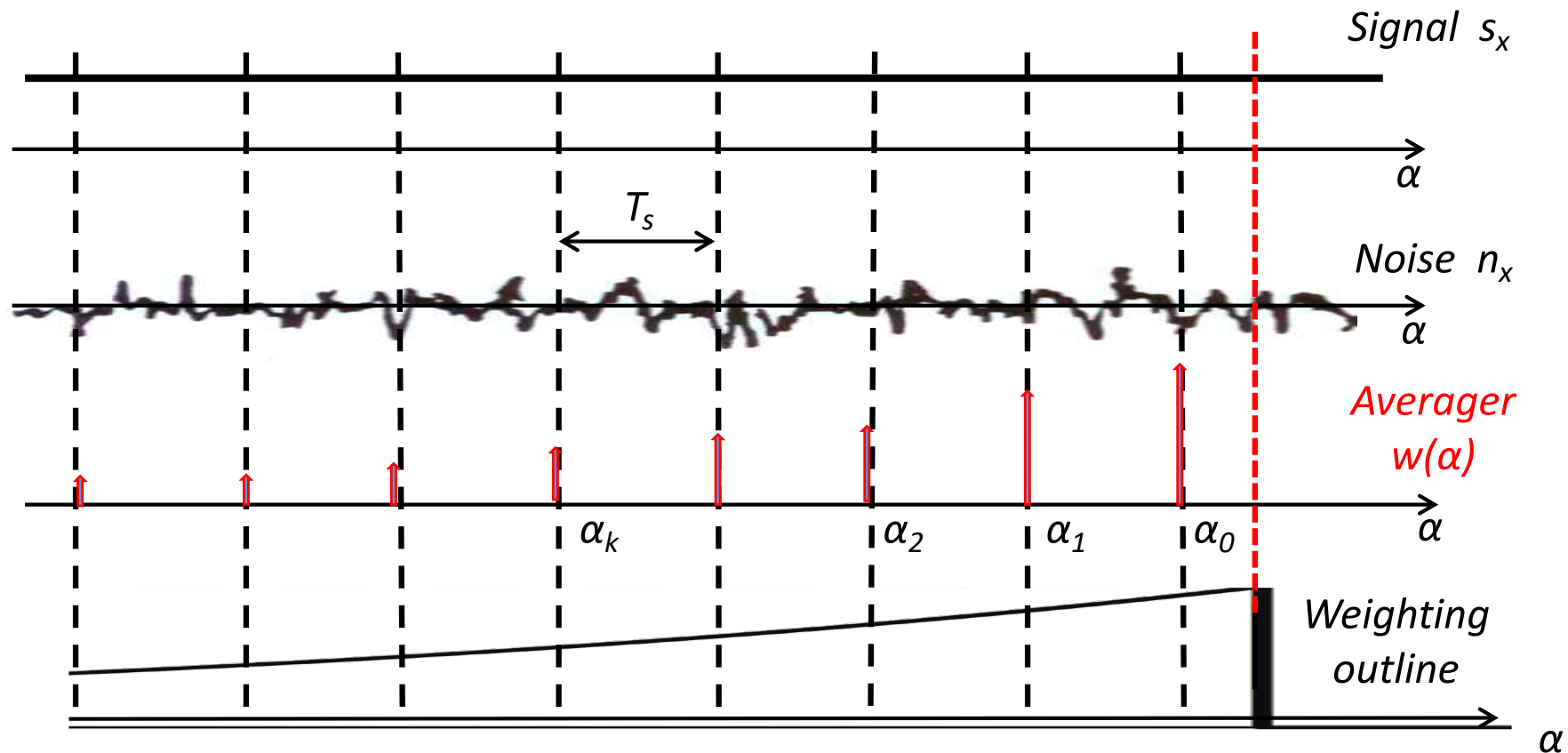$$\boldsymbol{s_y = s_x}$$

The output noise is reduced to

$$\overline{n_y{}^2} = N \cdot P^2 \overline{n_x{}^2} = \frac{1}{N} \cdot \overline{n_x{}^2}$$

$$\sqrt{\overline{\boldsymbol{n_y{}^2}}} = \frac{\sqrt{\overline{\boldsymbol{n_x{}^2}}}}{\sqrt{\boldsymbol{N}}}$$

which corresponds to the enhancement of the S/N

$$\left(\frac{S}{N}\right)_y = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

It is the discrete-time equivalent of an RC integrator

Signal s_x

α

T_s

Noise n_x

α

Averager
$w(\alpha)$

α_k α_2 α_1 α_0

α

Weighting outline

α

- Samples are taken with sampling frequency $f_s = 1/T_s$ i.e. at intervals T_s

- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n \ll T_s$)

- The sample **weight slowly decays** with the sample «age»: $w_k = Pr^k$ **with $(1-r) \ll 1$**

Discrete-Time Exponential Averager

Output signal $\quad s_y = s_x \cdot P \cdot \sum_{k=0}^{\infty} r^k = s_x \cdot P \frac{1}{1-r}$ $\quad$ (i.e, DC gain $G = P \frac{1}{1-r}$)

Output mean square noise

$$\overline{n_y{}^2} = P^2 (\overline{n_{x0}{}^2} + r^2 \cdot \overline{n_{x1}{}^2} + \cdots + r^{2k} \cdot \overline{n_{xk}{}^2} + \cdots + r^k r^j \cdot \overline{n_{xk} n_{xj}} + \cdots)$$

The noise samples are not correlated ($\overline{n_{xk} n_{xj}} = 0$ for $k{\neq}j$)
and the noise is stationary ($\overline{n_{x0}{}^2} = \overline{n_{x1}{}^2} = \cdots = \overline{n_x{}^2}$)
Therefore

$$\overline{n_y{}^2} = \overline{n_x{}^2} \cdot P^2 \left(1 + r^2 + \ldots + r^{2k} + \ldots \right) = \overline{n_x{}^2} \cdot P^2 \cdot \frac{1}{1-r^2}$$
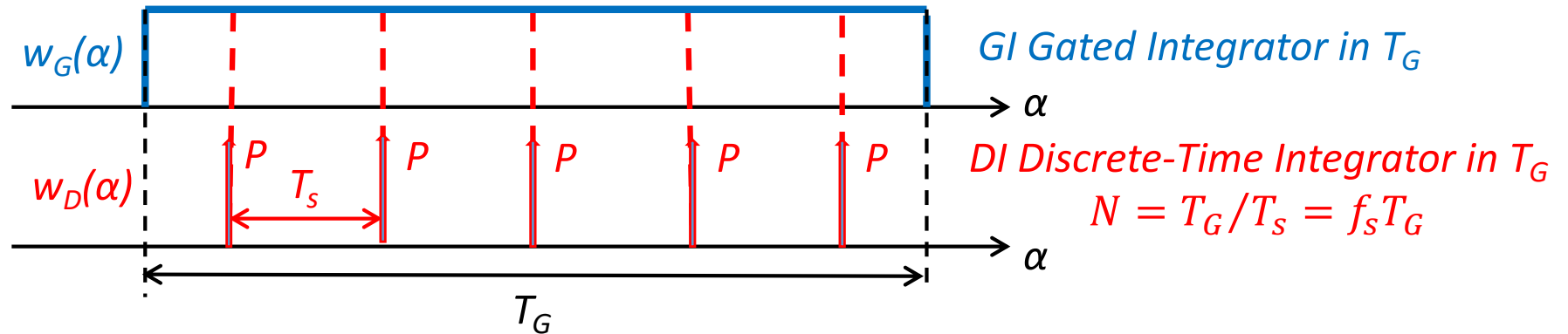
The SNR is thus improved to

$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y^2}}} = \frac{P s_x}{1-r} \frac{1}{\sqrt{\dfrac{\overline{n_x^2} P^2}{1-r^2}}} = \left(\frac{S}{N}\right)_x \sqrt{\frac{1+r}{1-r}}$$

But the attenuation ratio *r* is very close to unity (*1 − r) << 1* hence (*1 + r) ≈2* and therefore

$$\left(\frac{\boldsymbol{S}}{\boldsymbol{N}}\right)_{\boldsymbol{y}} \cong \left(\frac{\boldsymbol{S}}{\boldsymbol{N}}\right)_{\boldsymbol{x}} \sqrt{\frac{\boldsymbol{2}}{\boldsymbol{1-r}}}$$

Discrete-Time Integrator versus GI

Discrete-Time Integrator vs. GI

$w_G(\alpha)$ — *GI Gated Integrator in T_G*

$w_D(\alpha)$ — *DI Discrete-Time Integrator in T_G*
$$N = T_G/T_s = f_s T_G$$

INPUT: DC signal s_x and wide-band noise S_b (bandwidth $2f_n \gg f_s$, correlation width $2T_n \ll T_s$) with rms value $\overline{n_x^2} = S_b 2f_n = S_b/2T_n$

With unity DC gain $s_y = s_x$

- Noise reduction by GI $\qquad \sqrt{\overline{n_{yG}^2}} = \sqrt{\overline{n_x^2}} \Big/ \sqrt{\dfrac{T_G}{2T_n}}$

- Noise reduction by DI $\qquad \sqrt{\overline{n_{yD}^2}} = \sqrt{\overline{n_x^2}} \Big/ \sqrt{N}$

The improvement factor is

- $\sqrt{N}$ for the DI, increasing with the number N of samples taken

- $\sqrt{T_G/2T_n}$ for the GI, constant for a given T_G

QUESTION : is it possible to attain with a DI better S/N improvement than a GI just by increasing the number N (i.e. by using very fast sampling electronics)?

ANSWER: NO !!

In fact , since $N = T_G/T_s$ for having $N > T_G/2T_n$ it must be

$$T_s < 2T_n$$

in these conditions

- the samples are no more uncorrelated

- the improvement factor is **no more given by** $\sqrt{N}$

- There is still an improvement factor, but it must be evaluated taking into account the correlation between the noise samples.

- It is anyway $(S/N)_{DI} \leq (S/N)_{GI}$ with $(S/N)_{DI} \rightarrow (S/N)_{GI}$ as N is increased, as we can demonstrate in time domain and in frequency domain
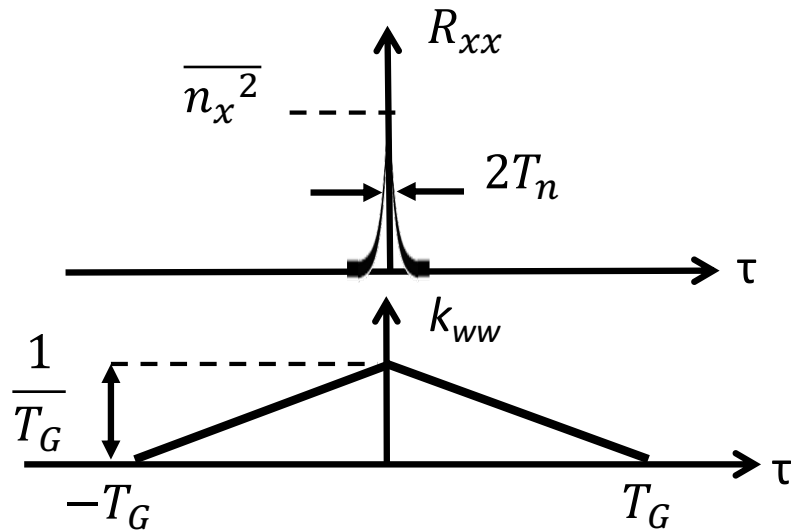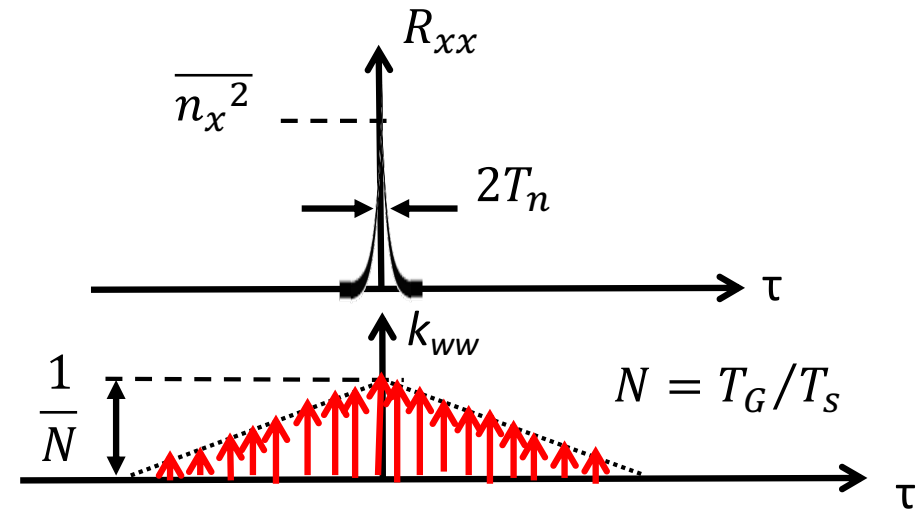
Discrete-Time Integrator vs. GI (time domain)

GI Gated Integrator
(normalized to unity DC gain G=1)

DI Discrete-time Integrator
(normalized to unity DC gain G=1)

R_{xx}

$\overline{n_x^2}$

$2T_n$

τ

k_{ww}

$\dfrac{1}{T_G}$

$-T_G$

T_G

τ

R_{xx}

$\overline{n_x^2}$

$2T_n$

τ

k_{ww}

$\dfrac{1}{N}$

$N = T_G/T_S$

τ

ZOOM around τ = 0

$k_{ww}(0) = \dfrac{1}{T_G}$

R_{xx}

τ

$\overline{n_x^2}$

R_{xx}

T_S

$k_{ww}(0) = \dfrac{1}{N} = \dfrac{T_S}{T_G}$

τ

Discrete-Time Integrator vs. GI (time domain)
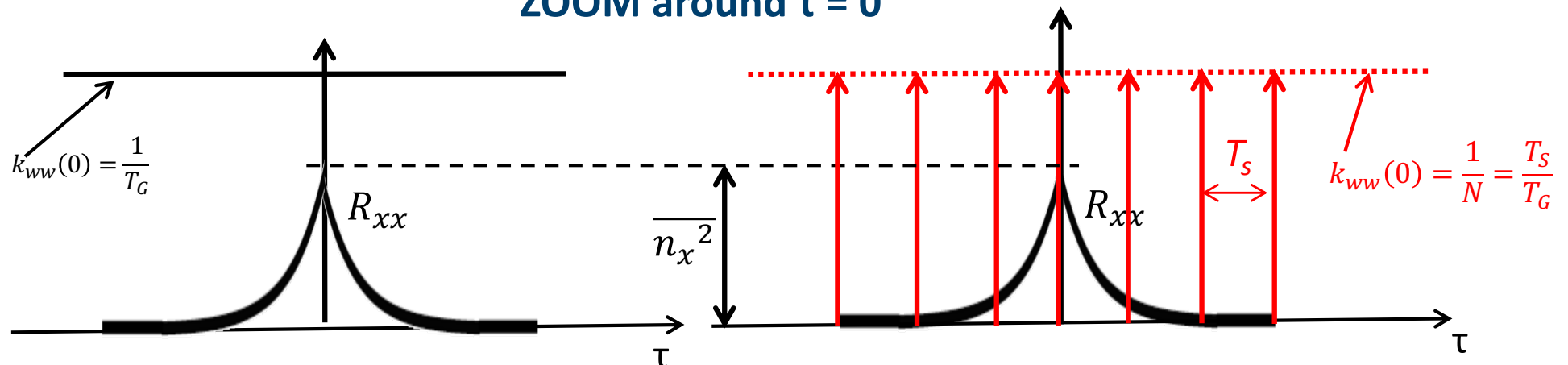
GI Gated Integrator (with G=1)

DI Discrete-time Integrator (with G=1)

$k_{ww}(0) = \dfrac{1}{T_G}$

R_{xx}

$\overline{n_x^2}$

$k_{ww}(0) = \dfrac{T_S}{T_G}$

T_S

τ

τ

$$\overline{n_{yD}^2} = \frac{1}{T_G} \cdot T_S \cdot (sum\ of\ R_{xx}\ samples\ at\ \tau = 0;\ \pm T_S;\ \pm 2T_S; ....)$$
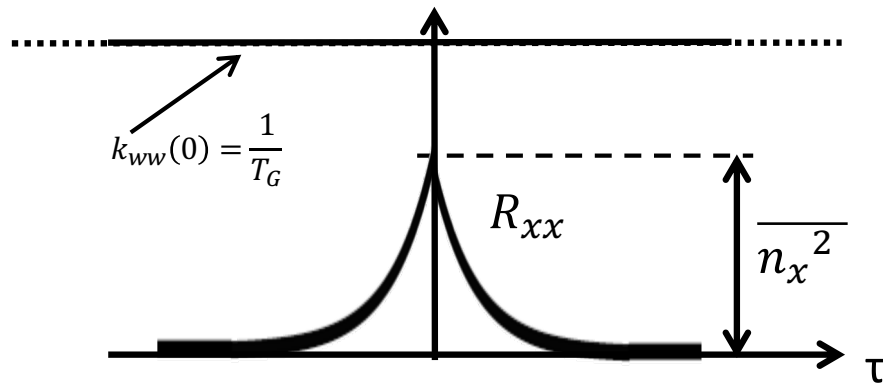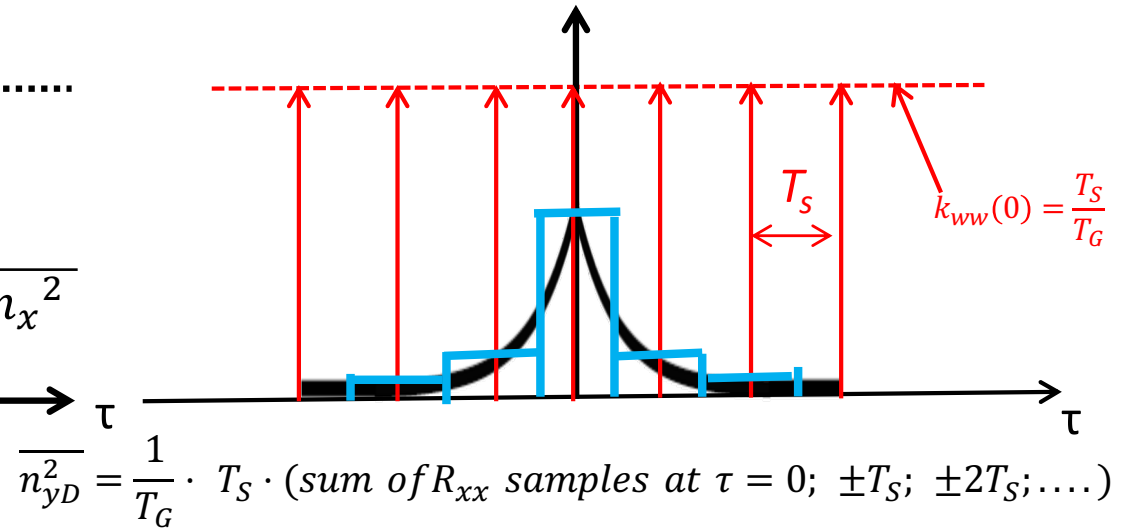
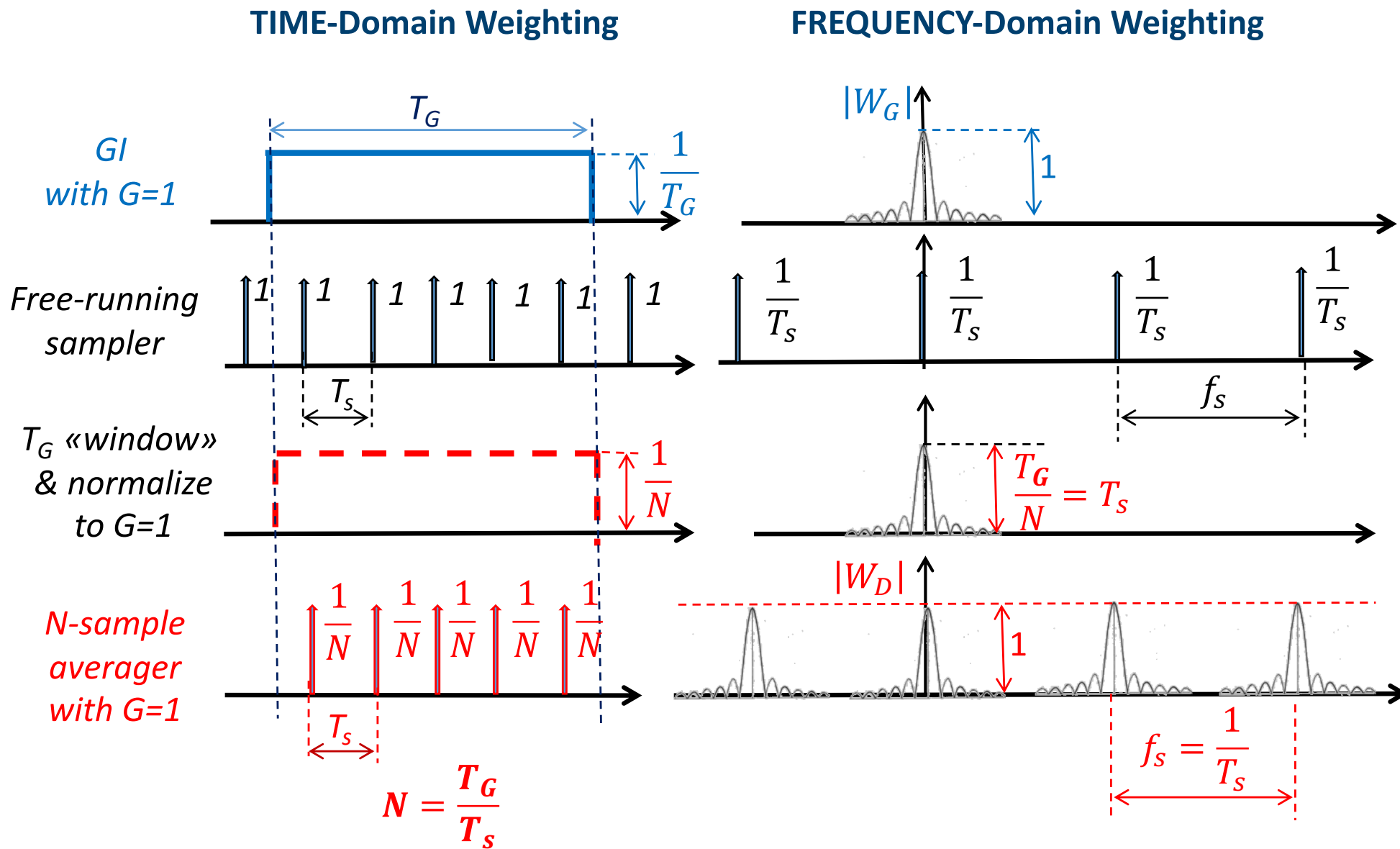$$\overline{n_{yG}^2} = \frac{1}{T_G} \cdot (area\ of\ R_{xx})$$

$$\overline{n_{yD}^2} = \frac{1}{T_G} \cdot (area\ of\ the\ scaloid\ that\ approximates\ R_{xx})$$
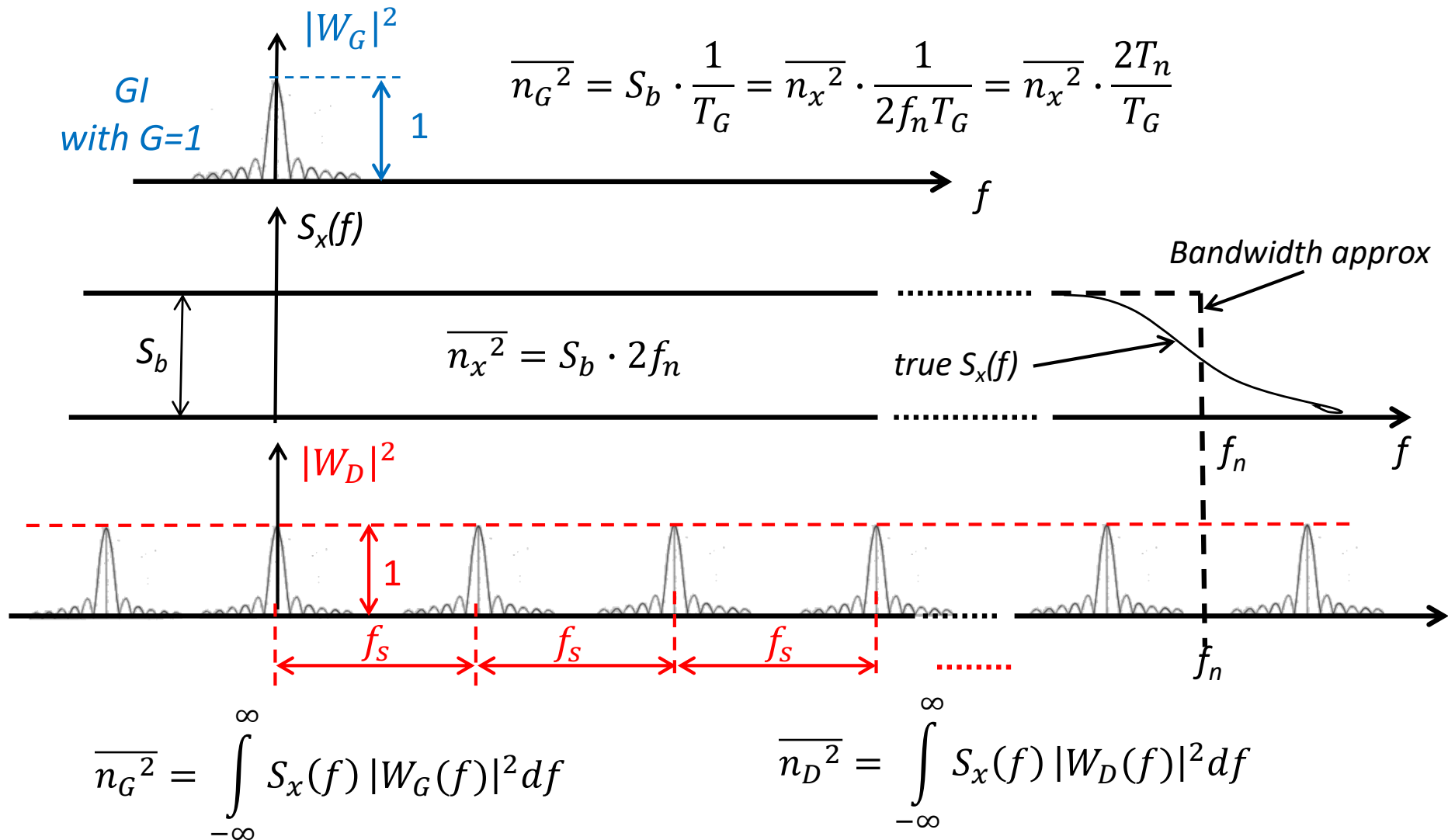
The scaloid area is greater than the R_{xx} area, therefore

$$\overline{n_{yD}^2} \geq \overline{n_{yG}^2} = \overline{n_x^2} \cdot \frac{2T_n}{T_G}$$

with

$$\overline{n_{yD}^2} \rightarrow \overline{n_{yG}^2} \quad as \quad T_S \rightarrow 0$$

Discrete-Time Integrator vs. GI (frequency domain)

TIME-Domain Weighting **FREQUENCY-Domain Weighting**

GI with G=1

T_G

$\dfrac{1}{T_G}$

$|W_G|$

1

Free-running sampler

$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$

T_s

$\dfrac{1}{T_s} \quad \dfrac{1}{T_s} \quad \dfrac{1}{T_s} \quad \dfrac{1}{T_s}$

f_s

T_G «window» & normalize to G=1

$\dfrac{1}{N}$

$\dfrac{T_G}{N} = T_s$

N-sample averager with G=1

$\dfrac{1}{N} \quad \dfrac{1}{N} \quad \dfrac{1}{N} \quad \dfrac{1}{N} \quad \dfrac{1}{N}$

T_s

$N = \dfrac{T_G}{T_s}$

$|W_D|$

1

$f_s = \dfrac{1}{T_s}$

Noise filtering analysis: GI vs. DI (frequency domain)

15

$|W_G|^2$

GI
with G=1

1

$$\overline{n_G^2} = S_b \cdot \frac{1}{T_G} = \overline{n_x^2} \cdot \frac{1}{2f_n T_G} = \overline{n_x^2} \cdot \frac{2T_n}{T_G}$$

f

$S_x(f)$

Bandwidth approx

S_b

$$\overline{n_x^2} = S_b \cdot 2f_n$$

true $S_x(f)$

f_n

f

$|W_D|^2$

1

f_s f_s f_s .... f_n

$$\overline{n_G^2} = \int_{-\infty}^{\infty} S_x(f) \, |W_G(f)|^2 df$$

$$\overline{n_D^2} = \int_{-\infty}^{\infty} S_x(f) \, |W_D(f)|^2 df$$

The figure illustrates how the output noise $\overline{n_D^2}$ is reduced and S/N is enhanced by increasing the sampling frequency f_s (for a given averaging time T_G)

a) **As long as** $f_s \ll f_n$:
- the noise samples are uncorrelated
- each line of $|W_D|^2$ is identical to $|W_G|^2$ of the GI (with same DC gain G=1)
- a high number N_L of lines of $|W_D|^2$ falls within the noise bandwidth $2f_n$
- the output noise of the DI is N_L times that of the GI

$$\overline{n_D^2} = \overline{n_G^2} \cdot N_L$$

With good approximation it is

$$N_L \cong 2f_n/f_s$$

and it is confirmed that for uncorrelated samples the S/N increases as $\sqrt{N}$

$$\overline{n_D^2} = \overline{n_x^2} \cdot \frac{1}{T_G f_s} = \frac{\overline{n_x^2}}{N}$$

b) **When f_s becomes comparable to f_n or higher**
- the previous result is no more valid.
- the output noise must be computed with the actual noise spectrum

$$\overline{n_D^2} = \int_{-\infty}^{\infty} S_x(f) |W_D(f)|^2 df \geq \overline{n_G^2}$$

- The figure shows that $\overline{n_D^2}$ is always higher than $\overline{n_G^2}$ and attains it for $f_s \to \infty$

$$\lim_{f_s \to \infty} \overline{n_D^2} = \overline{n_G^2}$$

Boxcar Integrator BI

Boxcar Integrator (BI)

This simple analog circuit combines two functions:

1. Sample Acquisition by gated integration
2. Exponential **averaging of samples**

The circuit employed is the same of the Gated Integrator, but with a fundamental difference: the capacitor is **NOT RESET between the acquisitions.**
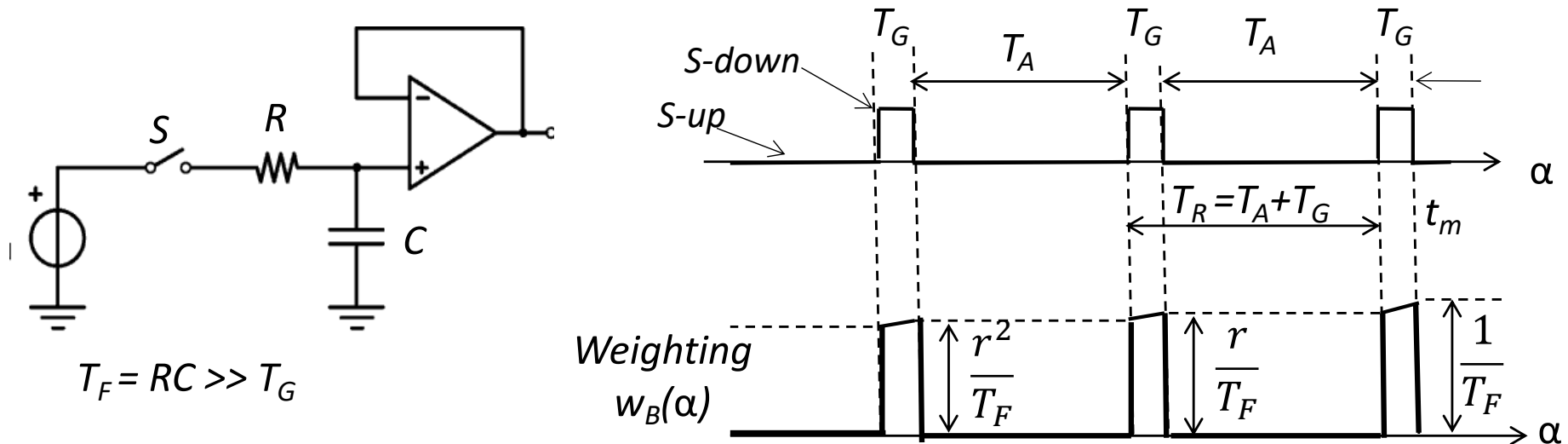
$$T_F = RC \gg T_G$$

- In T_A the C is in HOLD state: nothing changes, no memory loss and no new charge input
- In T_G the discharge of C (memory loss) reduces the previously stored value by the factor $r = e^{-T_G/T_F}$. **NB:** r does NOT depend on the interval T_A
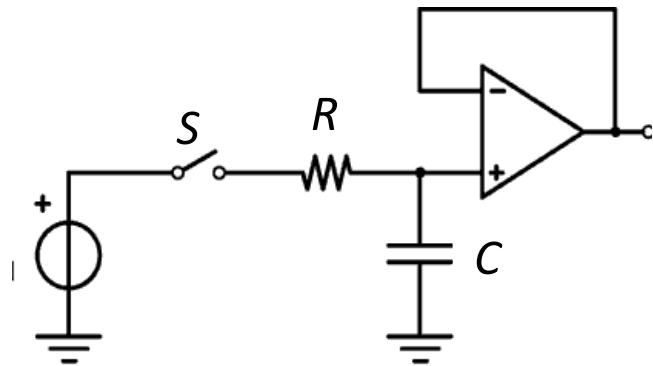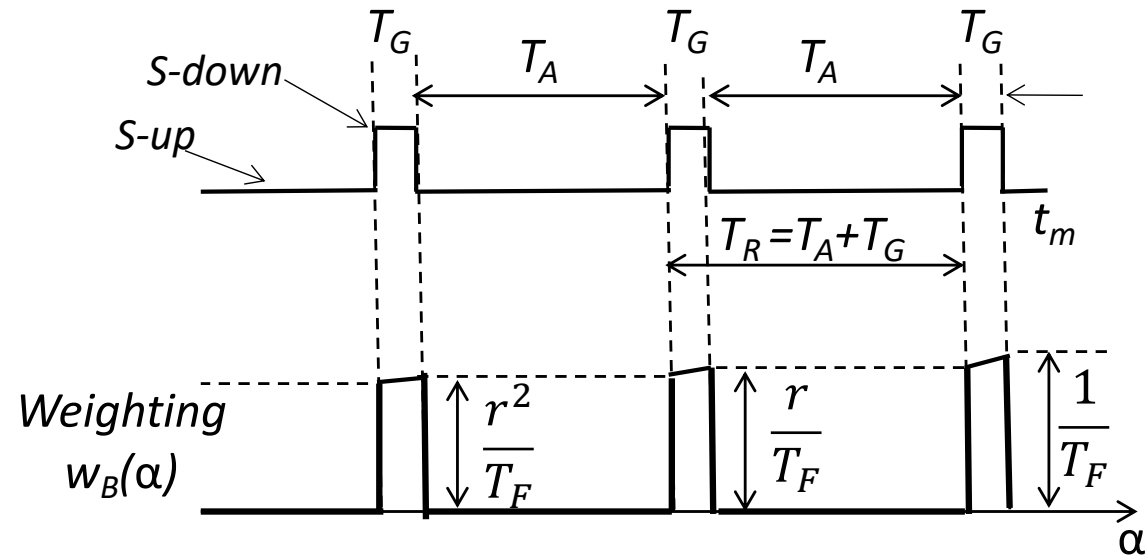
- BI behaves as RC-integrator (RCI) when the switch is closed (S-down); it is in HOLD state when the switch is open (S-up)

- In fact, the weighting function $w_B(\alpha)$ of the BI is obtained by subdividing $w_{RC}(\alpha)$ of the RCI it in «slices» of width T_G and placing them over the S-down intervals

- G=1 : the DC gain of BI (area of w_B) is unity (like that of RCI): the BI is an averager

- The autocorrelation functions k_{wwB} of BI and k_{wwRC} of RCI are very different, but have equal central value $k_{ww}(0)$

$$k_{wwB}(0) = k_{wwRC}(0) = \frac{1}{2RC} = \frac{1}{2T_F}$$

The input wide-band noise S_b with bandwidth $2f_n$, autocorrelation width $2T_n$, has mean square value

$$\overline{n_x{}^2} = S_b \cdot \frac{1}{2T_n}$$

The BI output noise is

$$\overline{n_y{}^2} = S_b \cdot k_{wwB}(0) = S_b \cdot 1/2T_F = \overline{n_x{}^2} \cdot \frac{T_n}{T_F}$$

Therefore, since BI has G=1 the S/N enhancement is

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

The S/N enhancement does **NOT** depend **on the RATE** of the samples because it is obtained by **averaging over a given number** of samples and **not over a given time interval**. In fact, counting the samples (from the measurement time t_m and going backwards) the sample weight is reduced below 1/100 for sample number $> 4.6 T_F/T_G$, irrespective of the sample rate

The BI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the BI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio
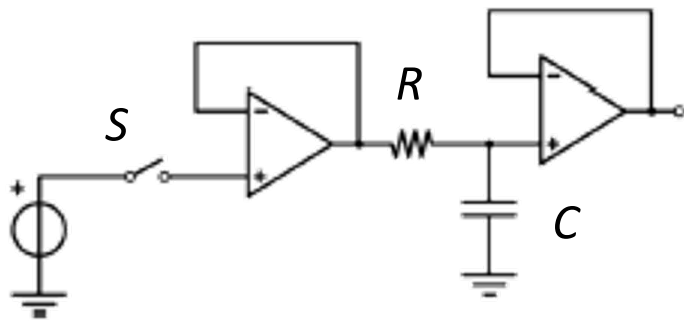$$r = e^{-T_G/T_F} \cong 1 - T_G/T_F$$
which enhances the S/N by the factor

$$\sqrt{(1+r)/(1-r)} \cong \sqrt{2/(1-r)} = \sqrt{2T_F/T_G}$$

NB: this factor is INDEPENDENT of the RATE of samples, because the AVERAGE IS DONE ON A GIVEN NUMBER OF SAMPLES and not on a given time.

The S/N enhancement is thus confirmed and clarified

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_F}{T_G}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

Ratemeter Integrator RI

Ratemeter Integrator (RI)

$T_R = RC \gg T_S$
for averaging many samples

$T_S = T_A + T_G$

S-down
S-up

T_G T_A T_G T_A T_G

RC-weighting $w_L(\alpha)$

$\dfrac{1}{T_R}$

RI-Weighting $w_R(\alpha)$

$\dfrac{r^2}{T_R}$ $\dfrac{r}{T_R}$ $\dfrac{1}{T_R}$
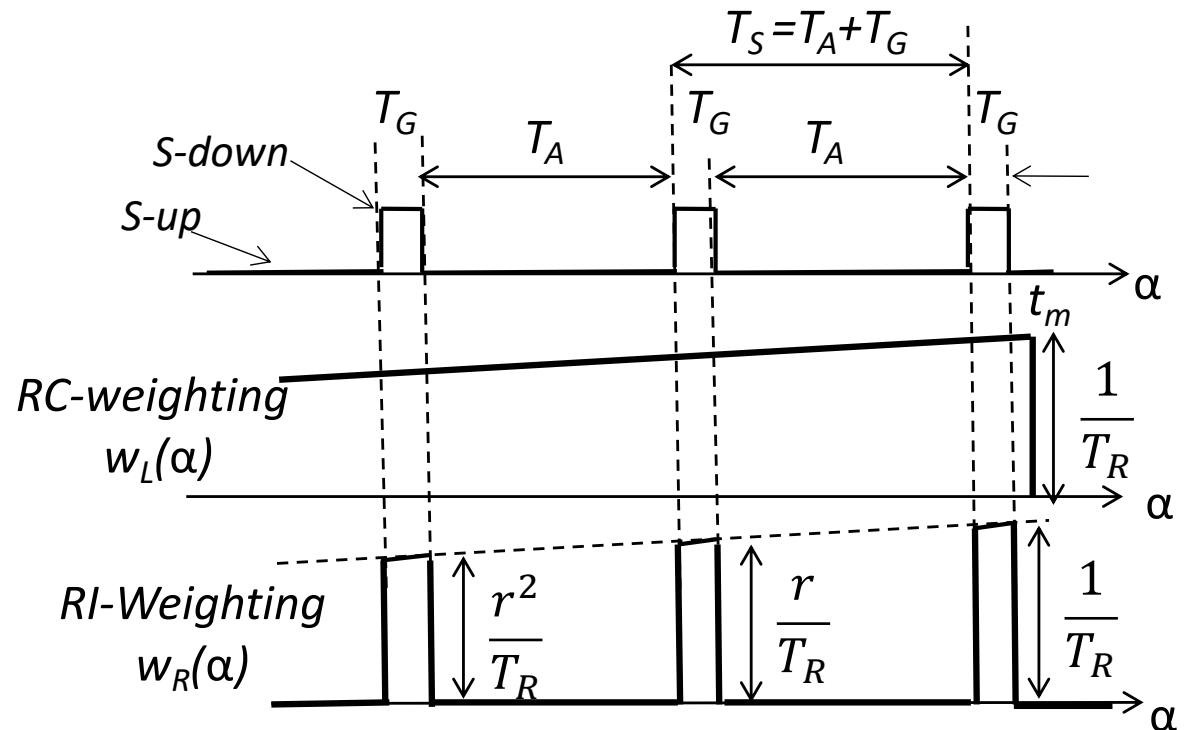
- By inserting a buffer between S and RC a new **exponential averager** is obtained, radically **different from BI**. The integrator is no more a switched-parameter RC filter: it is now a constant-parameter RC filter, unaffected by the switch S.

- There is no HOLD state. The memory loss goes on all the time; the weight reduction from sample to sample is $r = e^{-(T_G + T_A)/T_F} = e^{-T_S/T_F}$. **NB: *r* DEPENDS on the sample RATE!**

- During T_G (with S-down) the input is integrated in C
  During T_A (with S-up) the input is NOT allowed

Ratemeter Integrator (RI)
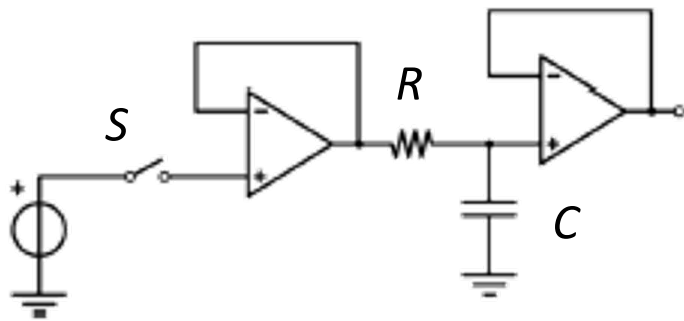
$T_R = RC \gg T_S$
for averaging many samples
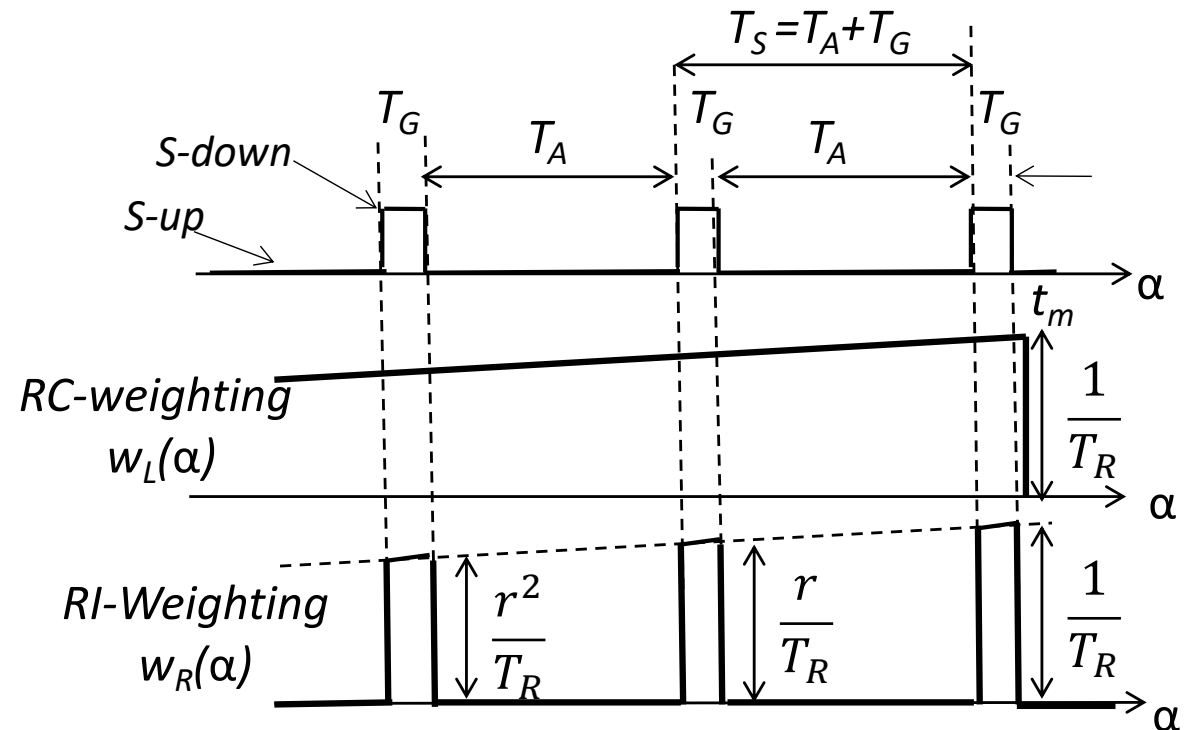
- The **DC gain is G < 1** (the RC filter has G=1, but it receives just a fraction of the input!)
- With $T_R \gg T_S$ the DC gain G is proportional to the sample rate $f_S = 1/T_S$

$$G = \int_{-\infty}^{\infty} w_R(\alpha)d\alpha \cong \frac{T_G}{T_S} \cdot \int_{-\infty}^{\infty} w_L(\alpha)d\alpha \cong \frac{T_G}{T_S} = f_S \cdot T_G$$

NB: if the input signal amplitude x_S is constant but f_S varies, the output signal y_S varies.
In fact, the circuit is also employed as **analog ratemeter**: with constant input voltage x_S
it produces a quasi DC output signal proportional to the repetition rate f_S

The RI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the RI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_S/T_R} \cong 1 - T_S/T_R$$

which enhances the S/N by the factor

$$\sqrt{\frac{1+r}{1-r}} \cong \sqrt{\frac{2}{1-r}} = \sqrt{\frac{2T_R}{T_S}} = \sqrt{2T_R f_S}$$

NB: this factor DEPENDS on the sample RATE f_S because the AVERAGE IS DONE ON A GIVEN TIME and not on a given number of samples. The weight reduction is below 1/100 for samples that at the measurement time t_m are «older» than $4.6 \cdot T_R$

The S/N enhancement thus depends on the sample rate f_s

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_R}{T_S}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{f_S T_G \frac{T_R}{T_n}}$$
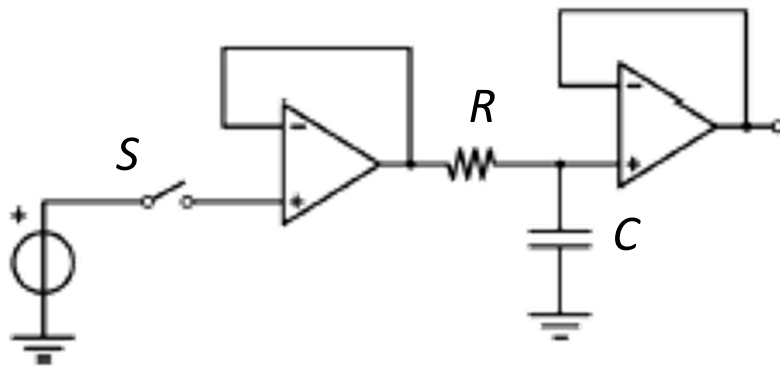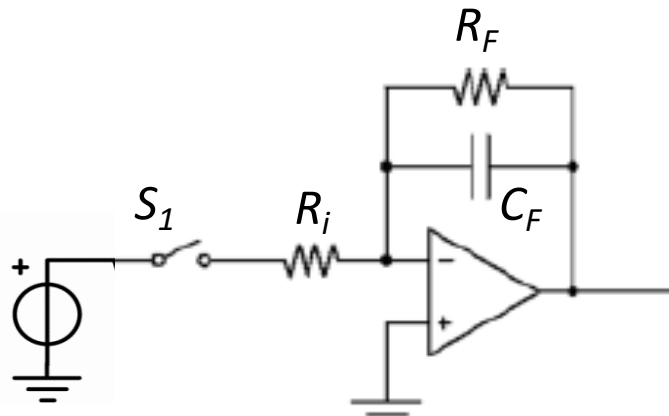
RATEMETER INTEGRATOR

- Switch S acts as gate on the input source

- Switch S is decoupled from the RC passive filter by the voltage buffer

- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state

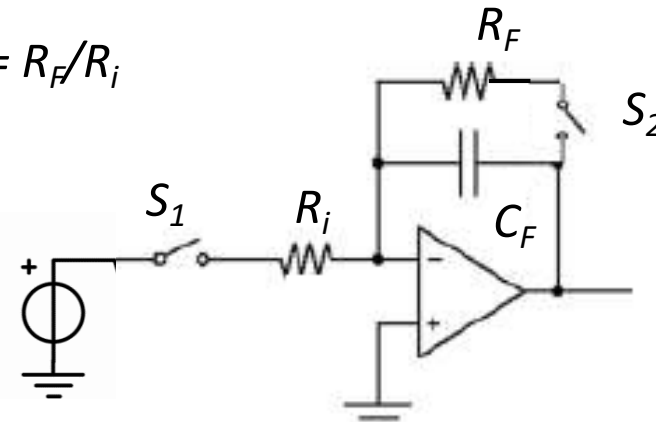- The sample average is done on a given time, defined by the RC value
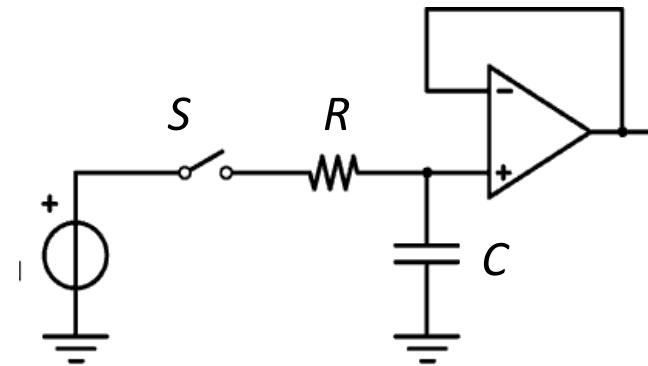
BOXCAR INTEGRATOR

- Switch S acts as gate on the input source

- Switch S acts also on the RC passive filter (changes the resistor value)

- The time constant T_F of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)

- The sample average is done on a given number of samples, defined by the T_F/T_G value

DC gain $G = R_F/R_i$

RATEMETER INTEGRATOR

- Switch S_1 acts as gate on the input

- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground

- The $R_F C_F$ integrator is unaffected by S_1; it has constant parameters, it does NOT have a HOLD state

- The sample average is done on a given time, defined by the $R_F C_F$ value

BOXCAR INTEGRATOR

- Switch S_1 acts as gate on the input

- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground

- A second switch S_2 is required for switching the time constant T_F of the integrator from finite $R_F C_F$ (S_2-down) to infinite (S_2-up, HOLD state)

- The sample average is done on a given number of samples, defined by the T_F/T_G value