

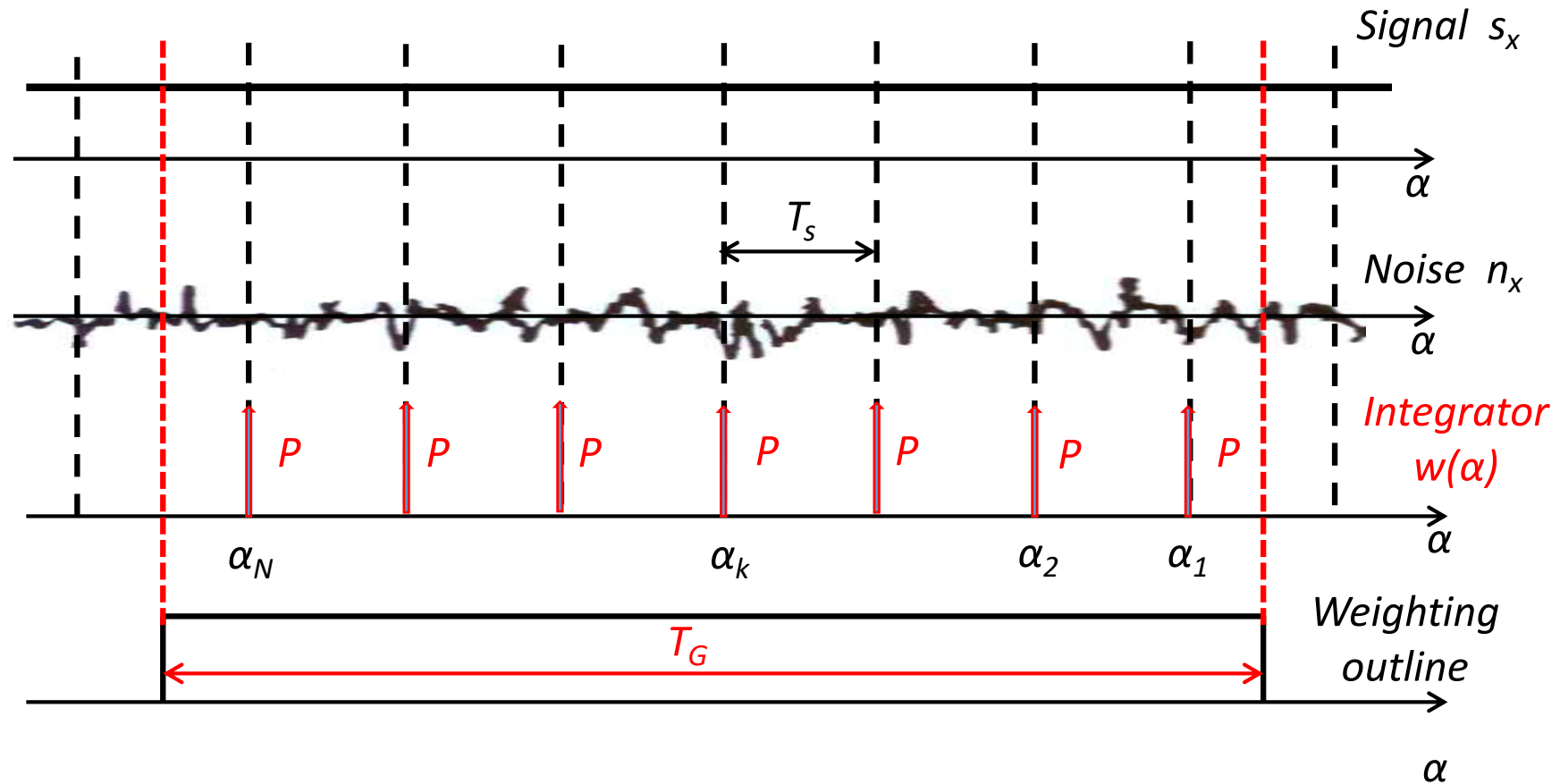
COURSE OUTLINE

- Introduction
- Signals and Noise
- **Filtering: LPF3 Switched-Parameter Averaging Filters**
- Sensors and associated electronics

- Discrete Time Integrator (DTI)
- Discrete Time Integrator versus Gated Integrator
- Boxcar Integrator (BI)
- Ratemeter Integrator (RI)

Discrete-Time Integrator DI

It is the **discrete-time** equivalent of a continuous gated integrator with gate $T_G = N T_s$



- Samples taken with sampling frequency $f_s = 1/T_s$ i.e. at intervals T_s within T_G
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n \ll T_s$)
- Every sample is multiplied by P and summed, up to a total $N = T_G / T_s$ samples

With **white noise**, the GI gives $S/N \propto \sqrt{T_G}$; which is the situation with a DI?

The **output signal** is

$$s_y = N \cdot P s_x \quad (\text{that is, the DC gain is } G = N \cdot P)$$

The **output noise** is $n_y = \sum_{k=1}^N P \cdot n_{xk}$ and

$$\overline{n_y^2} = P^2 (\overline{n_{x1}^2} + \overline{n_{x2}^2} \dots + \overline{n_{x1}n_{x2}} + \dots) = P^2 (\overline{n_{x1}^2} + \overline{n_{x2}^2} + \dots + \overline{n_{x1}n_{x2}} + \dots)$$

The noise samples are not correlated

$$\overline{n_{x1}n_{x2}} = \overline{n_{x2}n_{x3}} = \dots = 0$$

and the noise is stationary $\overline{n_{x1}^2} = \overline{n_{x2}^2} = \dots = \overline{n_x^2}$

Therefore

$$\overline{n_y^2} = N \cdot P^2 \overline{n_x^2}$$

By summing N samples the signal is increased by N and the rms noise by \sqrt{N}

The SNR is thus improved by the factor \sqrt{N}

$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y^2}}} = \frac{N \cdot P s_x}{\sqrt{N \cdot P^2 \overline{n_x^2}}} = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$



An averager is simply a discrete-time integrator with sampling weight P adjusted to give unity DC gain, that is $G = N \cdot P = 1$

$$P = \frac{1}{N}$$

and therefore output signal equal to input

$$s_y = s_x$$

The output noise is reduced to

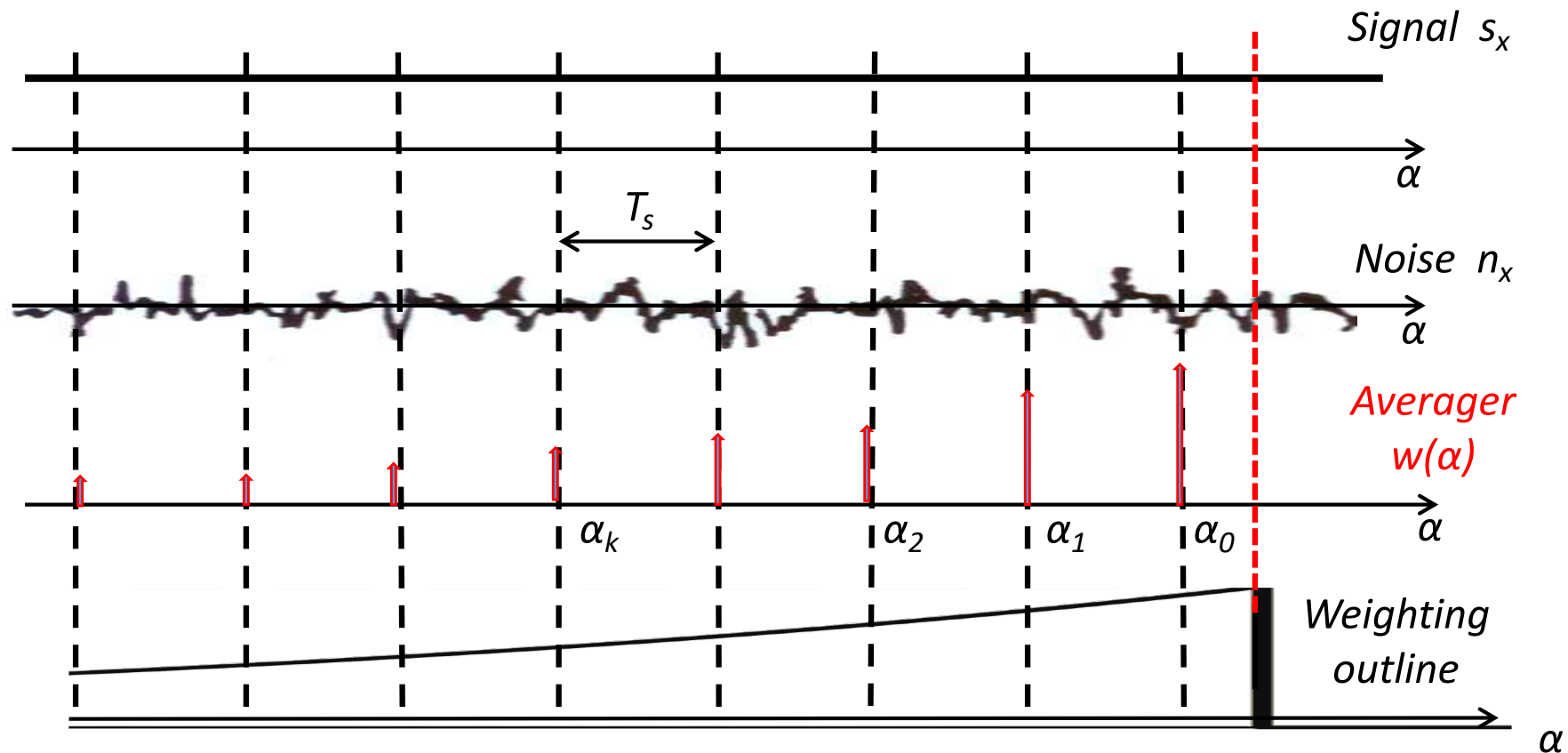
$$\overline{n_y^2} = N \cdot P^2 \overline{n_x^2} = \frac{1}{N} \cdot \overline{n_x^2}$$

$$\sqrt{\overline{n_y^2}} = \frac{\sqrt{\overline{n_x^2}}}{\sqrt{N}}$$

which corresponds to the enhancement of the S/N

$$\left(\frac{S}{N}\right)_y = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

It is the discrete-time equivalent of an RC integrator



- Samples are taken with sampling frequency $f_s = 1/T_s$ i.e. at intervals T_s
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n \ll T_s$)
- The sample **weight slowly decays** with the sample «age»: $w_k = Pr^k$ **with $(1 - r) \ll 1$**

Output signal $s_y = s_x \cdot P \cdot \sum_{k=0}^{\infty} r^k = s_x \cdot P \frac{1}{1-r}$ (i.e, DC gain $G = P \frac{1}{1-r}$)

Output mean square noise

$$\overline{n_y^2} = P^2 (\overline{n_{x0}^2} + r^2 \cdot \overline{n_{x1}^2} + \dots + r^{2k} \cdot \overline{n_{xk}^2} + \dots + r^k r^j \cdot \overline{n_{xk} n_{xj}} + \dots)$$

The noise samples are not correlated ($\overline{n_{xk} n_{xj}} = 0$ for $k \neq j$)

and the noise is stationary ($\overline{n_{x0}^2} = \overline{n_{x1}^2} = \dots = \overline{n_x^2}$)

Therefore

$$\overline{n_y^2} = \overline{n_x^2} \cdot P^2 (1 + r^2 + \dots + r^{2k} + \dots) = \overline{n_x^2} \cdot P^2 \cdot \frac{1}{1-r^2}$$

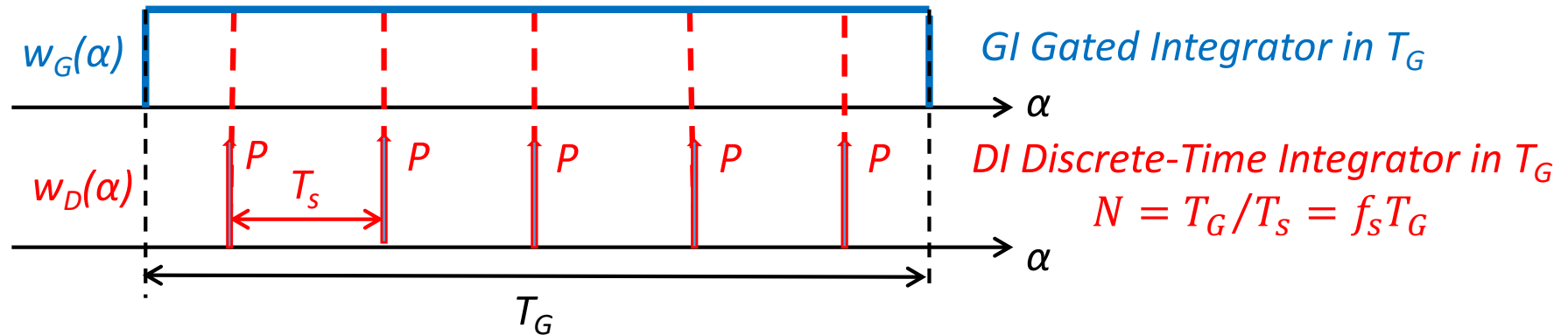
The SNR is thus improved to

$$\left(\frac{S}{N} \right)_y = \frac{s_y}{\sqrt{\overline{n_y^2}}} = \frac{P s_x}{1-r} \frac{1}{\sqrt{\frac{\overline{n_x^2} P^2}{1-r^2}}} = \left(\frac{S}{N} \right)_x \sqrt{\frac{1+r}{1-r}}$$

But the attenuation ratio r is very close to unity ($1-r \ll 1$) hence ($1+r \approx 2$) and therefore

$$\left(\frac{S}{N} \right)_y \cong \left(\frac{S}{N} \right)_x \sqrt{\frac{2}{1-r}} \quad \text{WITH CONSTANT INPUT}$$

Discrete-Time Integrator versus GI



INPUT: DC signal s_x and wide-band noise S_b (bandwidth $2f_n \gg f_s$, correlation width $2T_n \ll T_s$) with rms value $\overline{n_x^2} = S_b 2f_n = S_b / 2T_n$

With unity DC gain $s_y = s_x$

- Noise reduction by GI $\sqrt{\overline{n_{yG}^2}} = \sqrt{\overline{n_x^2}} / \sqrt{\frac{T_G}{2T_n}}$
- Noise reduction by DI $\sqrt{\overline{n_{yD}^2}} = \sqrt{\overline{n_x^2}} / \sqrt{N}$

The improvement factor is

- \sqrt{N} for the DI, increasing with the number N of samples taken
- $\sqrt{T_G/2T_n}$ for the GI, constant for a given T_G

QUESTION : is it possible to attain with a DI better S/N improvement than a GI just by increasing the number N (i.e. by using very fast sampling electronics)?

ANSWER: NO !!

In fact , since $N = T_G/T_s$ for having $N > T_G/2T_n$ it must be

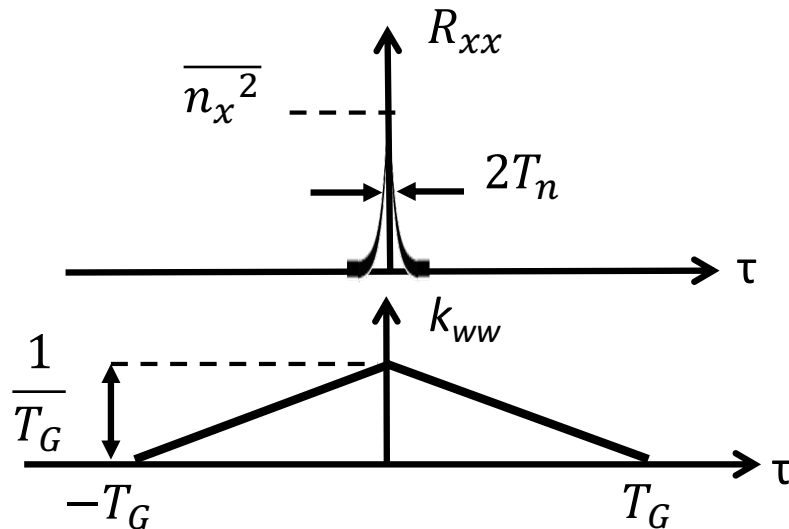
$$T_s < 2T_n$$

in these conditions

- the samples are no more uncorrelated
- the improvement factor is **no more given by \sqrt{N}**
- There is still an improvement factor, but it must be evaluated taking into account the correlation between the noise samples.
- It is anyway $(S/N)_{DI} \leq (S/N)_{GI}$ with $(S/N)_{DI} \rightarrow (S/N)_{GI}$ as N is increased, as we can demonstrate in time domain and in frequency domain

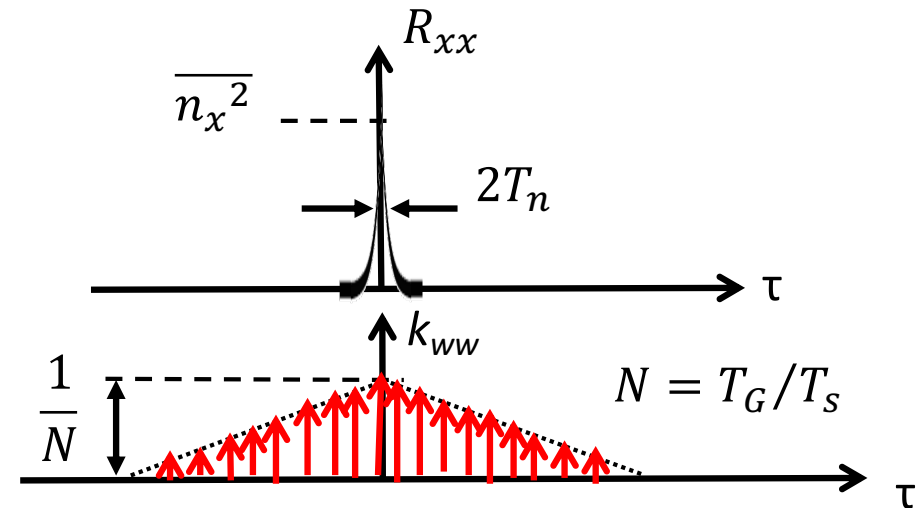
GI Gated Integrator

(normalized to unity DC gain $G=1$)

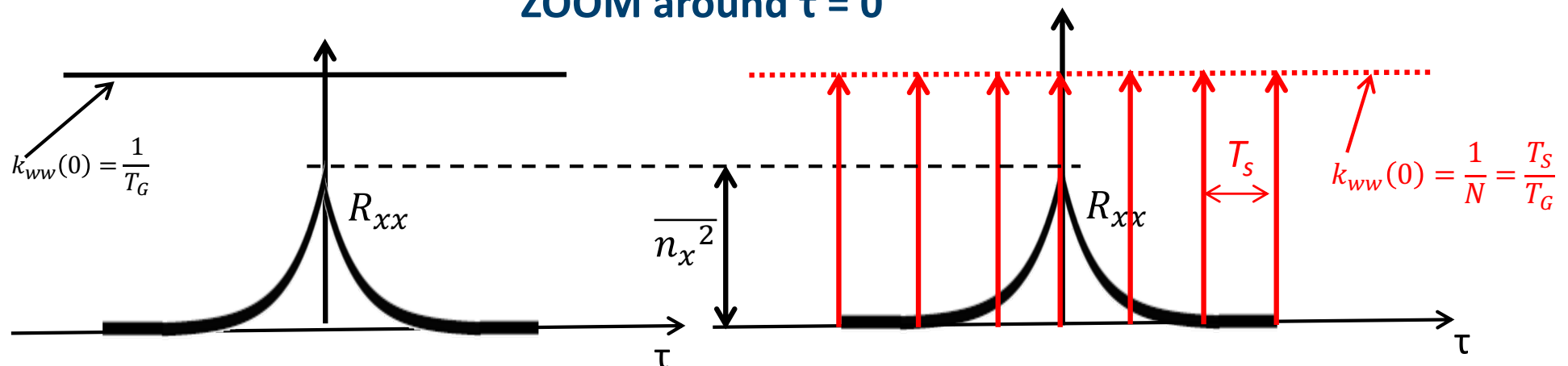


DI Discrete-time Integrator

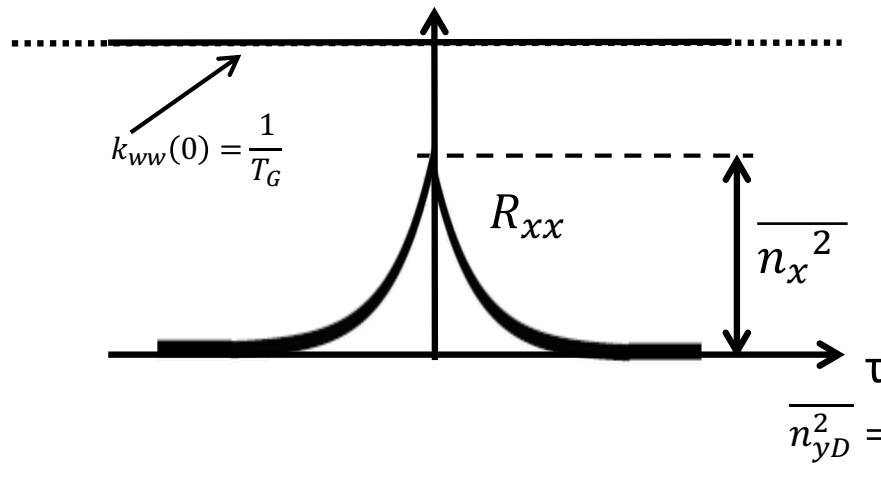
(normalized to unity DC gain $G=1$)



ZOOM around $\tau = 0$

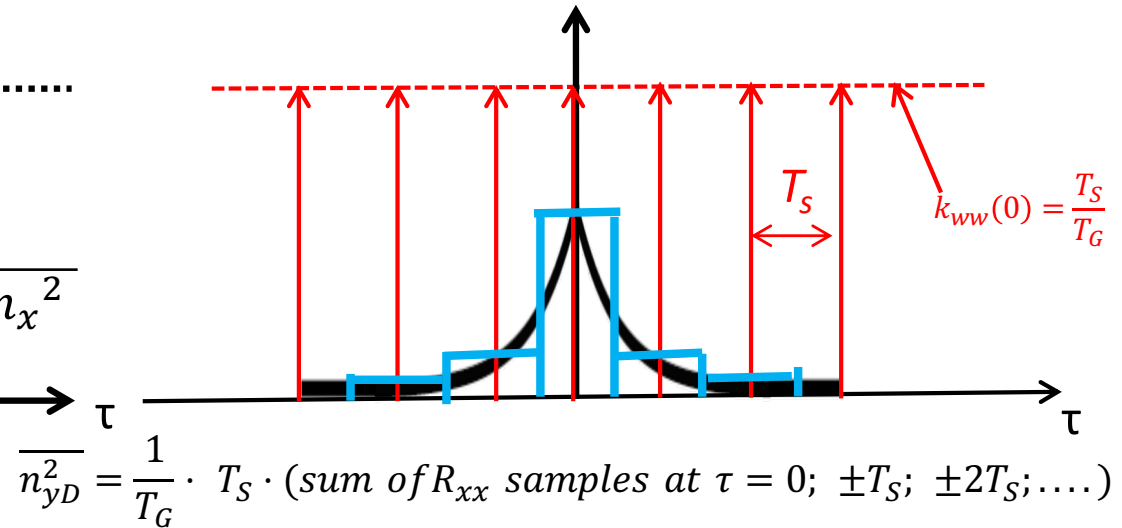


GI Gated Integrator (with G=1)



$$\overline{n_{yG}^2} = \frac{1}{T_G} \cdot (\text{area of } R_{xx})$$

DI Discrete-time Integrator (with G=1)



$$\overline{n_{yD}^2} = \frac{1}{T_G} \cdot (\text{area of the scaloid that approximates } R_{xx})$$

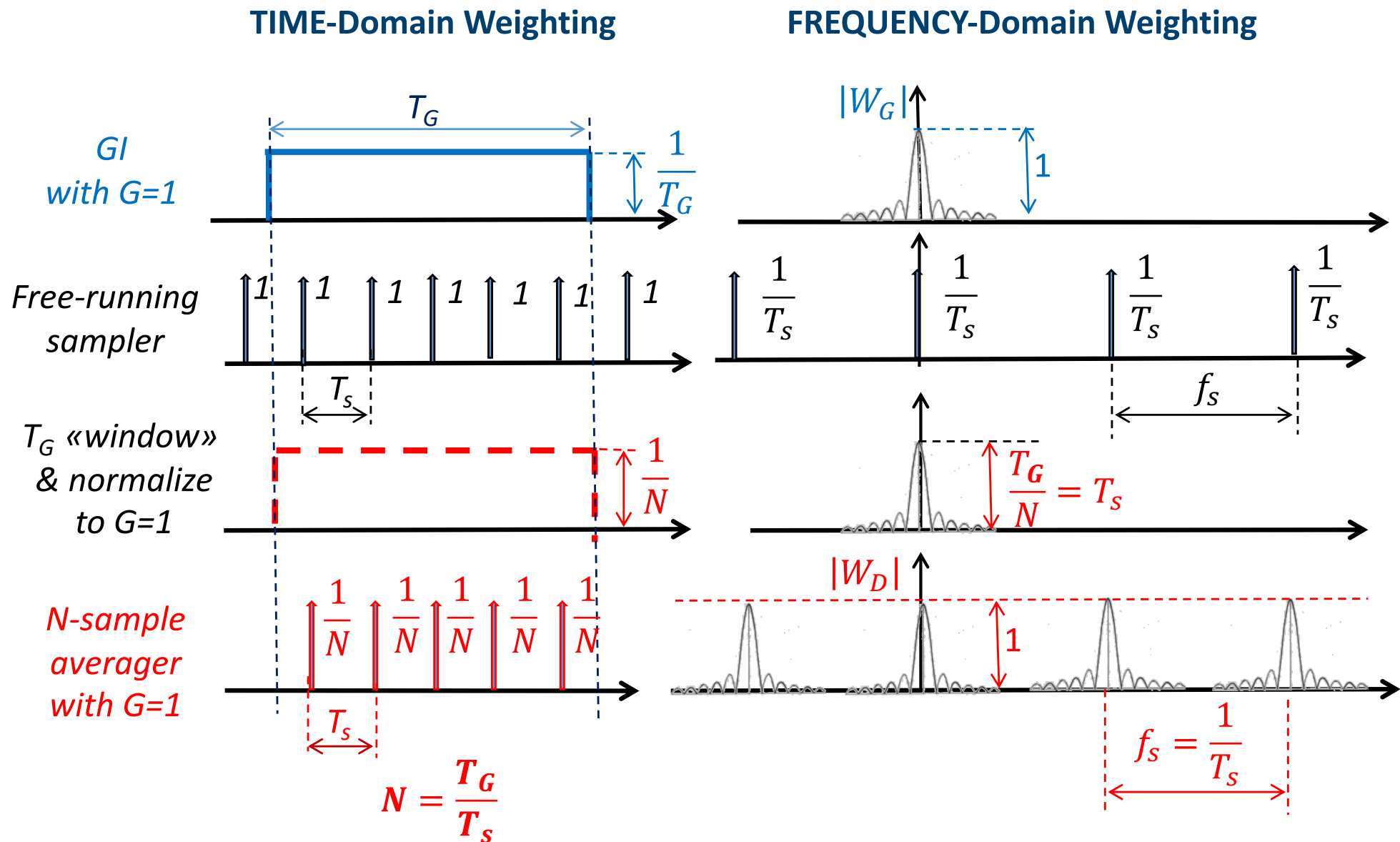
The scaloid area is greater than the R_{xx} area, therefore

$$\overline{n_{yD}^2} \geq \overline{n_{yG}^2} = \overline{n_x^2} \cdot \frac{2T_n}{T_G}$$

with

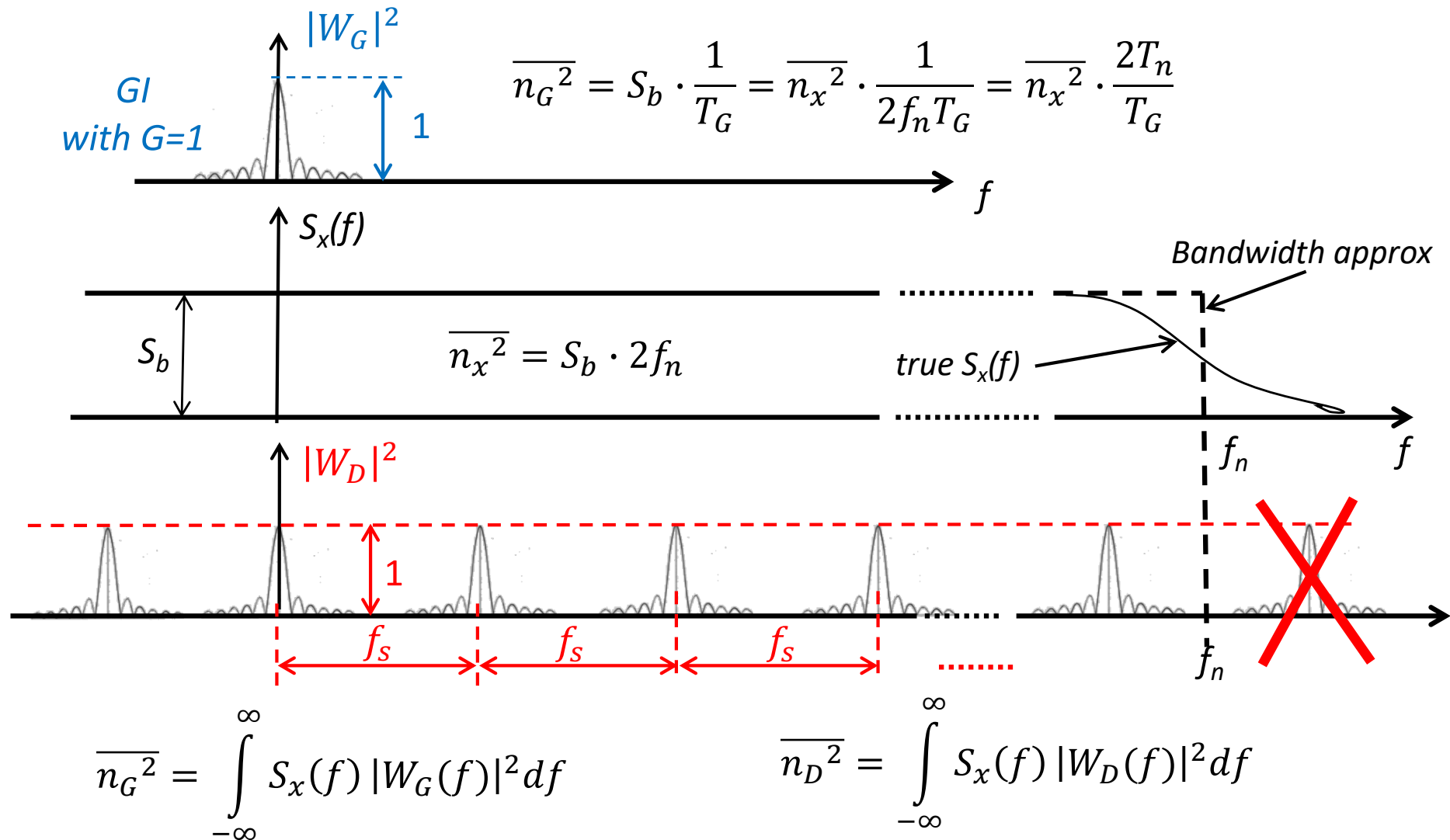
$$\overline{n_{yD}^2} \rightarrow \overline{n_{yG}^2} \quad \text{as} \quad T_S \rightarrow 0$$

Discrete-Time Integrator vs. GI (frequency domain)



Noise filtering analysis: GI vs. DI (frequency domain)

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The figure illustrates how the output noise $\overline{n_D^2}$ is reduced and S/N is enhanced by increasing the sampling frequency f_s (for a given averaging time T_G)

a) As long as $f_s \ll f_n$:

- the noise samples are uncorrelated
- each line of $|W_D|^2$ is identical to $|W_G|^2$ of the GI (with same DC gain $G=1$)
- a high number N_L of lines of $|W_D|^2$ falls within the noise bandwidth $2f_n$
- the output noise of the DI is N_L times that of the GI

$$\overline{n_D^2} = \overline{n_G^2} \cdot N_L$$

With good approximation it is

$$N_L \cong 2f_n/f_s$$

and it is confirmed that for uncorrelated samples the S/N increases as \sqrt{N}

$$\overline{n_D^2} = \overline{n_x^2} \cdot \frac{1}{T_G f_s} = \frac{\overline{n_x^2}}{N}$$

b) When f_s becomes comparable to f_n or higher

- the previous result is no more valid.
- the output noise must be computed with the actual noise spectrum

$$\overline{n_D^2} = \int_{-\infty}^{\infty} S_x(f) |W_D(f)|^2 df \geq \overline{n_G^2}$$

- The figure shows that $\overline{n_D^2}$ is always higher than $\overline{n_G^2}$ and attains it for $f_s \rightarrow \infty$

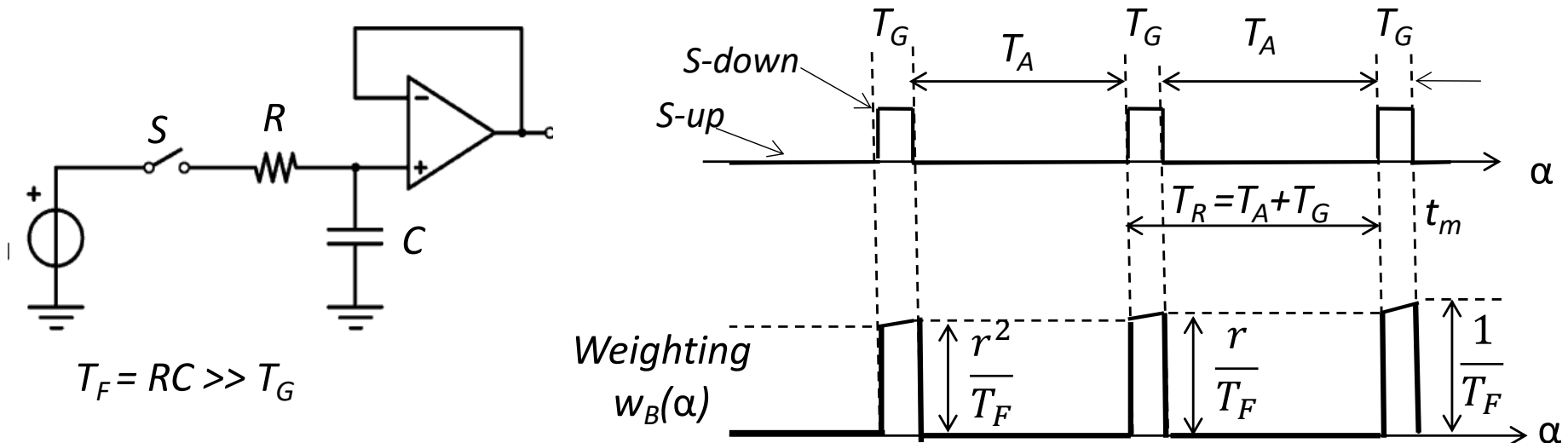
$$\lim_{f_s \rightarrow \infty} \overline{n_D^2} = \overline{n_G^2}$$

Boxcar Integrator BI

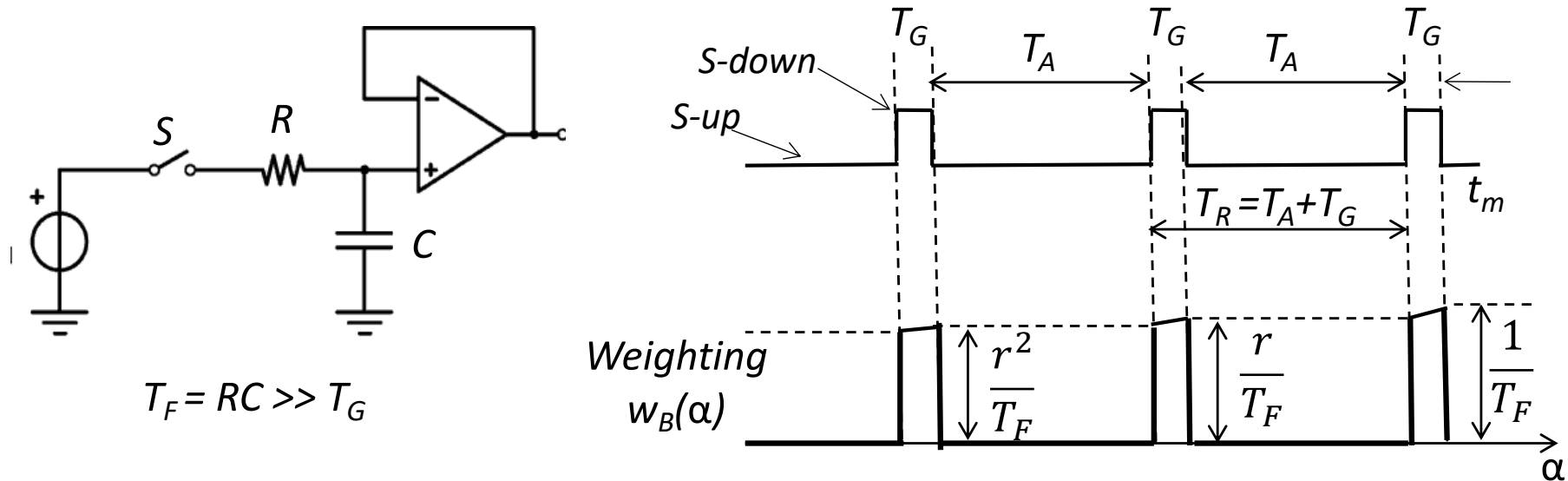
This simple analog circuit combines two functions:

1. Sample Acquisition by **gated integration**
2. Exponential **averaging of samples**

The circuit employed is the same of the Gated Integrator, but with a fundamental difference: the capacitor is **NOT RESET between the acquisitions.**



- In T_A the C is in HOLD state: nothing changes, no memory loss and no new charge input
- In T_G the discharge of C (memory loss) reduces the previously stored value by the factor $r = e^{-T_G/T_F}$. **NB:** r does NOT depend on the interval T_A



- BI behaves as RC-integrator (RCI) when the switch is closed (S-down); it is in HOLD state when the switch is open (S-up)
- In fact, the weighting function $w_B(\alpha)$ of the BI is obtained by subdividing $w_{RC}(\alpha)$ of the RCI in «slices» of width T_G and placing them over the S-down intervals
- $G=1$: the DC gain of BI (area of w_B) is unity (like that of RCI): the BI is an averager
- The autocorrelation functions k_{wwB} of BI and k_{wwRC} of RCI are very different, but have equal central value $k_{ww}(0)$

$$k_{wwB}(0) = k_{wwRC}(0) = \frac{1}{2RC} = \frac{1}{2T_F}$$

The input wide-band noise S_b with bandwidth $2f_n$, autocorrelation width $2T_n$, has mean square value

$$\overline{n_x^2} = S_b \cdot \frac{1}{2T_n}$$

The BI output noise is

$$\overline{n_y^2} = S_b \cdot k_{wwB}(0) = S_b \cdot 1/2T_F = \overline{n_x^2} \cdot \frac{T_n}{T_F}$$

Therefore, since BI has $G=1$ the S/N enhancement is

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$



The S/N enhancement does **NOT** depend **on the RATE** of the samples because it is obtained by **averaging over a given number** of samples and **not over a given time interval**.

The BI is equivalent to the cascade of two filtering stages

- a) Acquisition of samples by a GI with same T_G and T_F as the BI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

- b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_G/T_F} \cong 1 - T_G/T_F$$

which enhances the S/N by the factor

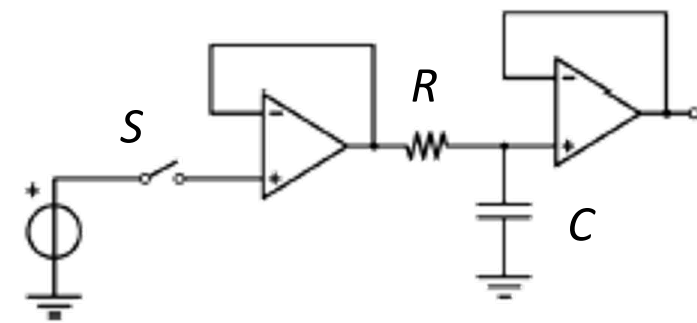
$$\sqrt{(1+r)/(1-r)} \cong \sqrt{2/(1-r)} = \sqrt{2T_F/T_G}$$

NB: this factor is INDEPENDENT of the RATE of samples, because the AVERAGE IS DONE ON A GIVEN NUMBER OF SAMPLES and not on a given time.

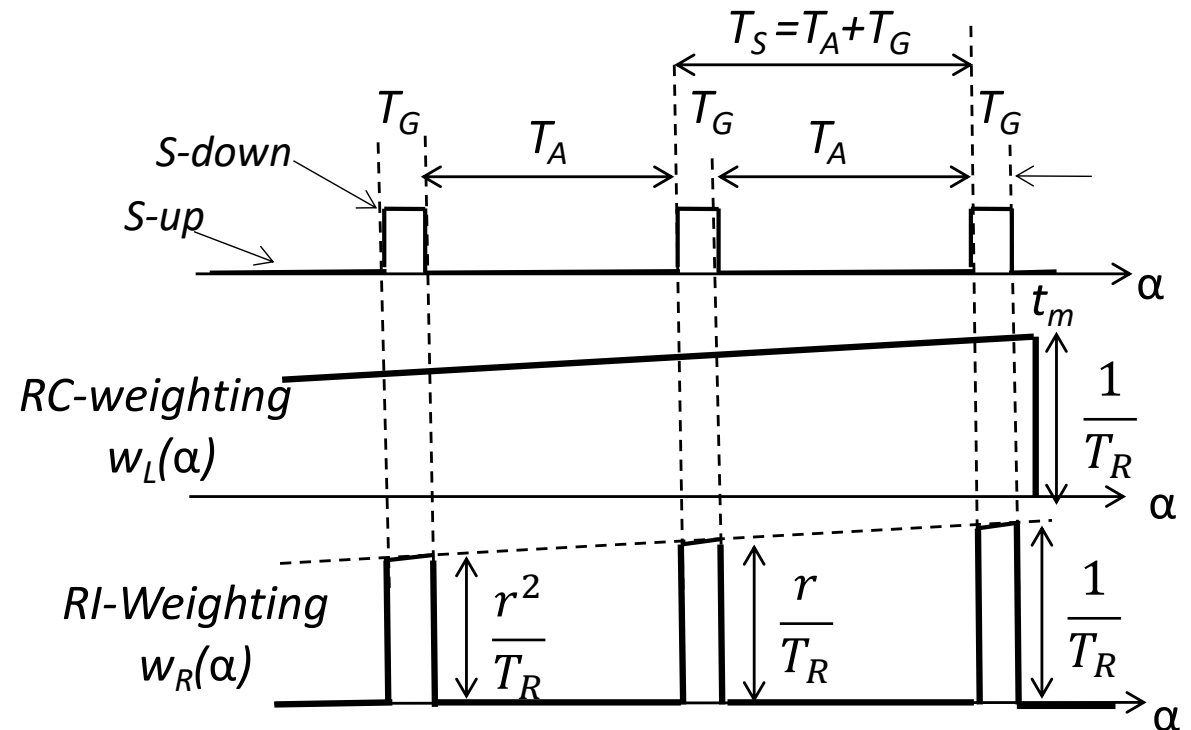
The S/N enhancement is thus confirmed and clarified

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_F}{T_G}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

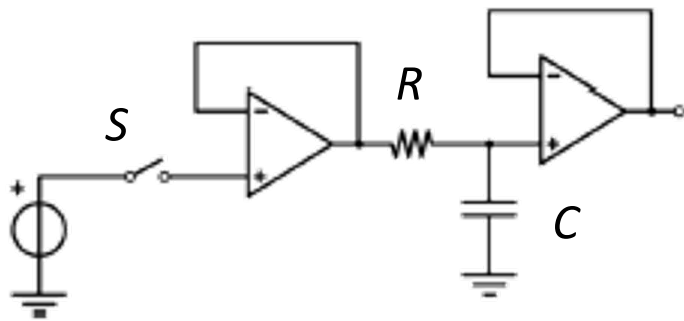
Ratemeter Integrator RI



$T_R = RC \gg T_S$
for averaging many samples

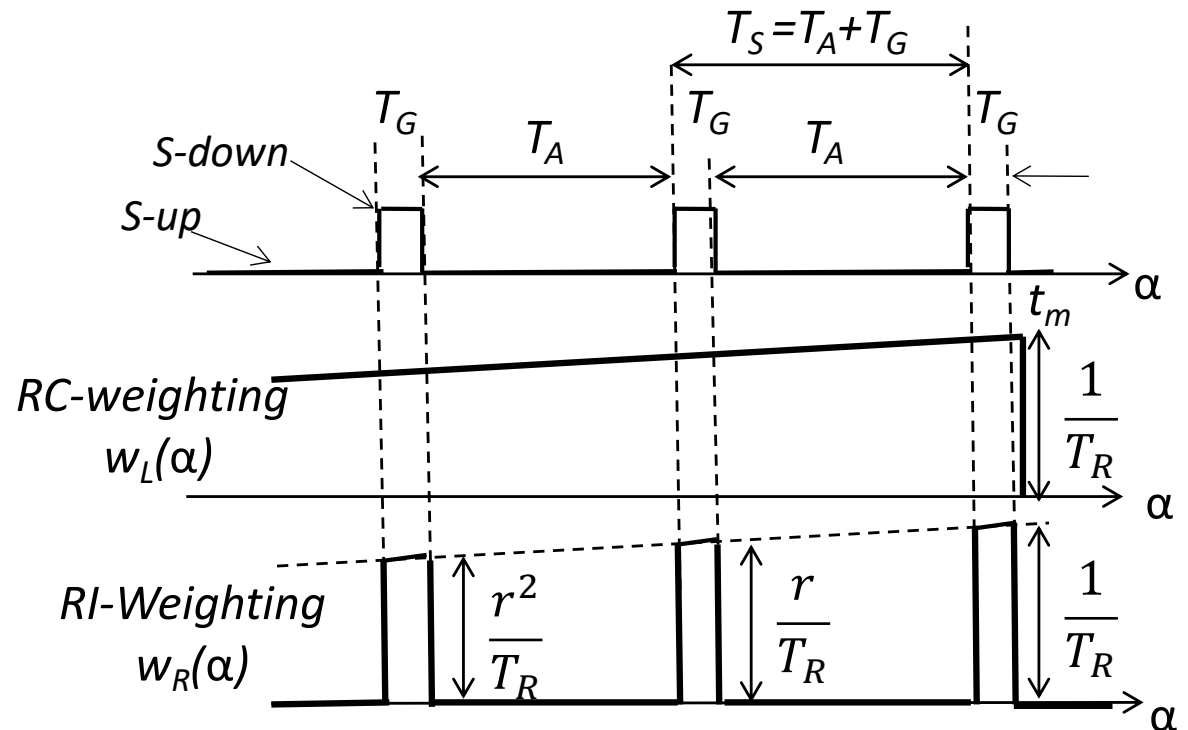


- By inserting a buffer between S and RC a new **exponential averager** is obtained, radically **different from BI**. The integrator is no more a switched-parameter RC filter: it is now a constant-parameter RC filter, unaffected by the switch S.
- There is no HOLD state. The memory loss goes on all the time; the weight reduction from sample to sample is $r = e^{-(T_G+T_A)/T_F} = e^{-T_S/T_F}$. **NB: r DEPENDS on the sample RATE!**
- During T_G (with S-down) the input is integrated in C
During T_A (with S-up) the input is NOT allowed



$$T_R = RC \gg T_S$$

for averaging many samples



- The **DC gain is $G < 1$** (the RC filter has $G=1$, but it receives just a fraction of the input!)
- With $T_R \gg T_S$ the DC gain G is proportional to the sample rate $f_S = 1/T_S$

$$G = \int_{-\infty}^{\infty} w_R(\alpha) d\alpha \cong \frac{T_G}{T_S} \cdot \int_{-\infty}^{\infty} w_L(\alpha) d\alpha \cong \frac{T_G}{T_S} = f_S \cdot T_G$$

NB: if the input signal amplitude x_S is constant but f_S varies, the output signal y_S varies. In fact, the circuit is also employed as **analog ratemeter**: with constant input voltage x_S it produces a quasi DC output signal proportional to the repetition rate f_S

The RI is equivalent to the cascade of two filtering stages

- a) Acquisition of samples by a GI with same T_G and T_F as the RI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

- b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_S/T_R} \cong 1 - T_S/T_R$$

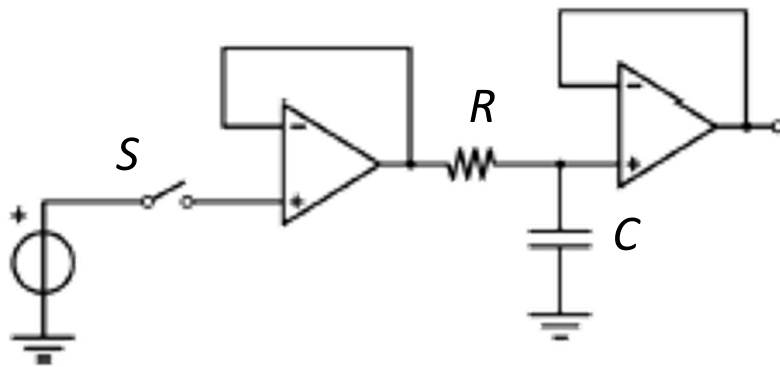
which enhances the S/N by the factor

$$\sqrt{\frac{1+r}{1-r}} \cong \sqrt{\frac{2}{1-r}} = \sqrt{\frac{2T_R}{T_S}} = \sqrt{2T_R f_S}$$

NB: this factor DEPENDS on the sample RATE f_S because the AVERAGE IS DONE ON A GIVEN TIME and not on a given number of samples. The weight reduction is below 1/100 for samples that at the measurement time t_m are «older» than $4.6 \cdot T_R$

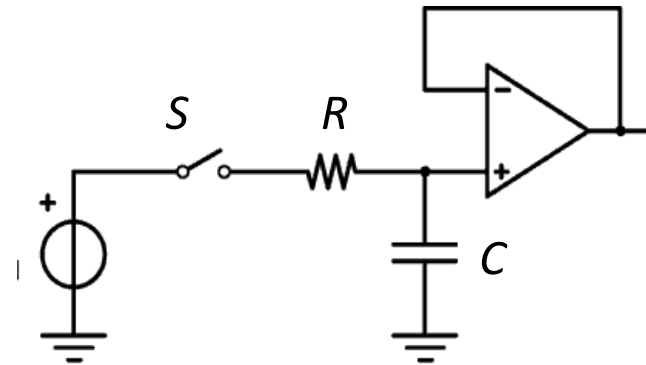
The S/N enhancement thus depends on the sample rate f_S

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_R}{T_S}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{f_S T_G \frac{T_R}{T_n}}$$



RATEMETER INTEGRATOR

- Switch S acts as gate on the input source
- Switch S is decoupled from the RC passive filter by the voltage buffer
- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the RC value



BOXCAR INTEGRATOR

- Switch S acts as gate on the input source
- Switch S acts also on the RC passive filter (changes the resistor value)
- The time constant T_F of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value