COURSE OUTLINE

- Introduction
- Signals and Noise
- **Filtering: LPF1 Constant-Parameter Low Pass Filters**
- Sensors and associated electronics

Constant-Parameter Low-Pass Filters

- Low-Pass Filters as Basic Elements for Signal and Noise Filtering
- RC Integrator
- Mobile-Mean Low-Pass Filter
- Bandwidth and Correlation Time of Low-Pass Filters

Low-Pass Filters as Basic Elements for Signal and Noise Filtering

Filtering signals and noise

SIGNALS AND NOISE

- Signals carry information, but are accompanied by noise
- The noise often is non-negligible and can degrade or even obscure the information
- Filtering is intended to improve the recovering of the information
- Filtering must **exploit** at best the **differences** between signal and noise, taking well into account **what kind of information** is to be recovered. For instance: in case of a pulse-signal, is it just the amplitude or is it the complete waveform?

LOW-PASS FILTERS

- We deal first with «low-pass filters» (LPF), so called because of their action in the frequency domain. The filtering weight is concentrated in a **relatively narrow frequency band** from zero to a limit frequency; above the band-limit it falls to negligible value.
- Correspondingly, in the time domain the weighting function has **relatively wide time-width** (as well as its autocorrelation).
- The action of the filter as seen in time-domain is to produce approximately a **time-average** (i.e. a weighted average) of the input over a finite time interval, delimited by the width of the weighting function

Low-pass filters LPF

To understand and to be able to deal with LPF is very important because:

a) LPF are a **basic element** of filtering and a **foundation** for gaining a better insight on all other kinds of filters and better exploit them.

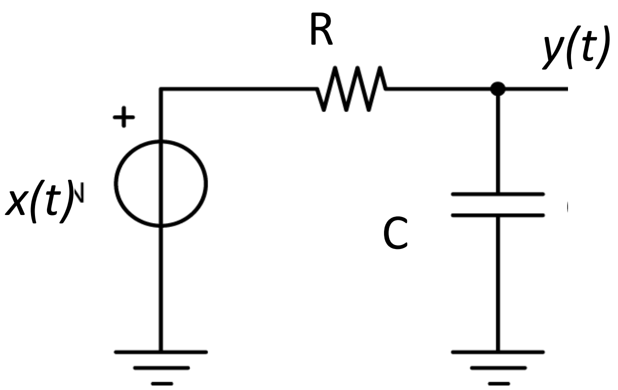
For instance, a high-pass filter (HPF) can be obtained by subtracting from a given input the output of an LPF that receives the same input. In various real HPF, the physical structure of the HPF actually implements this scheme.

b) LPF are **employed in real cases** of filtering for information recovery

For instance, in many cases a wide-band noise accompanies signals that have significant frequency components in a relatively narrow frequency band around $f=0$.
These are not only the cases of DC and slowly varying signals, but also cases where the just the **amplitude** of a **pulse signal** (having fairly long pulse-duration and known pulse shape) must be measured (and not the complete waveform)

RC-integrator

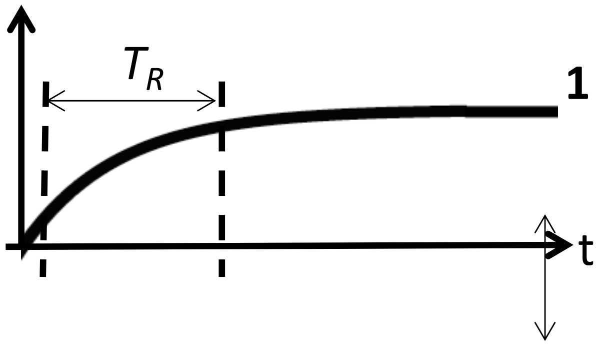
RC integrator (constant-parameter LPF)



$$T_f = RC$$

δ-response

$$h(t) = \frac{1}{T_f} 1(t) e^{-t/T_f}$$

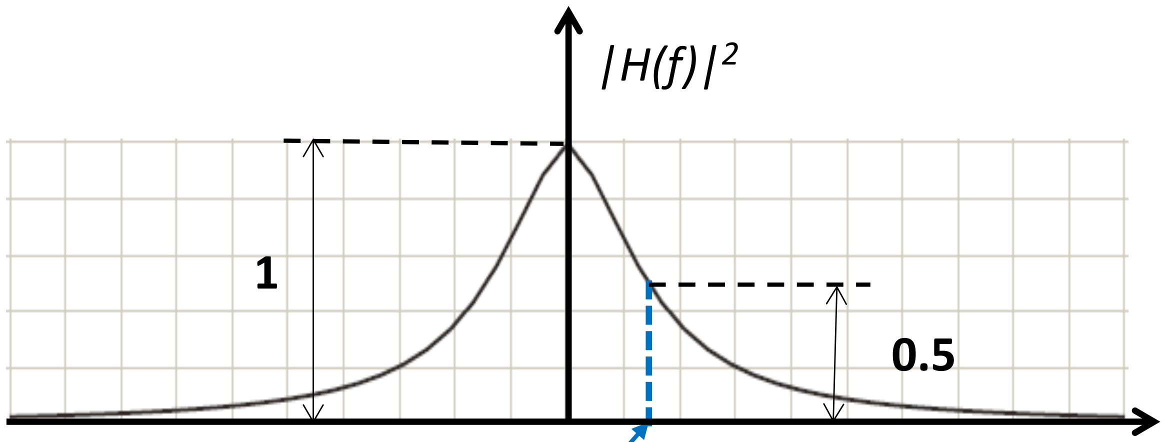


Step-response
with
risetime 10-90%
$T_R = 2.2\ T_f \approx 1/3f_p$

Transfer function

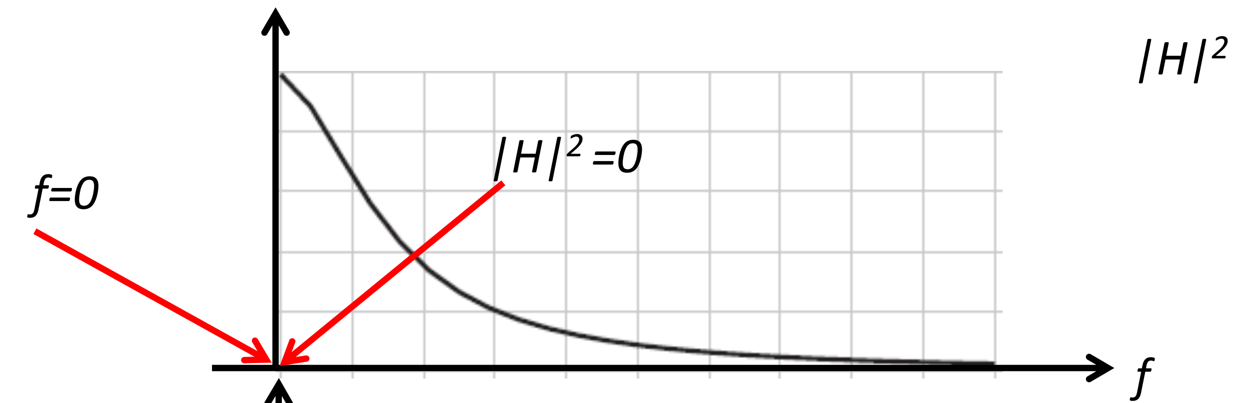
$$\boldsymbol{H(f) = \frac{1}{1 + j2\pi f T_f}}$$

$$|H(f)|^2 = \frac{1}{1 + (2\pi f T_f)^2}$$

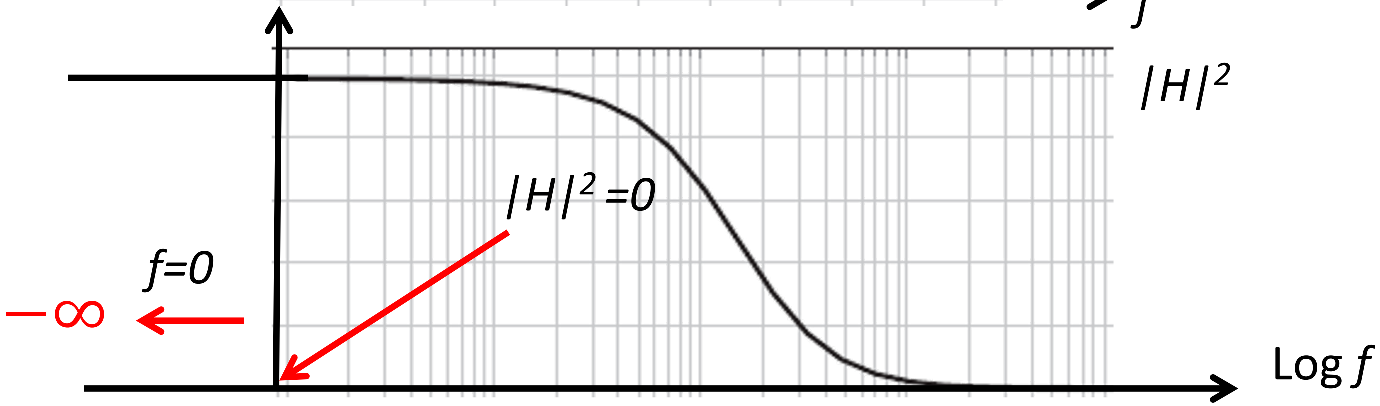


3 dB frequency (Pole) $f_p = 1/2\pi T_f$

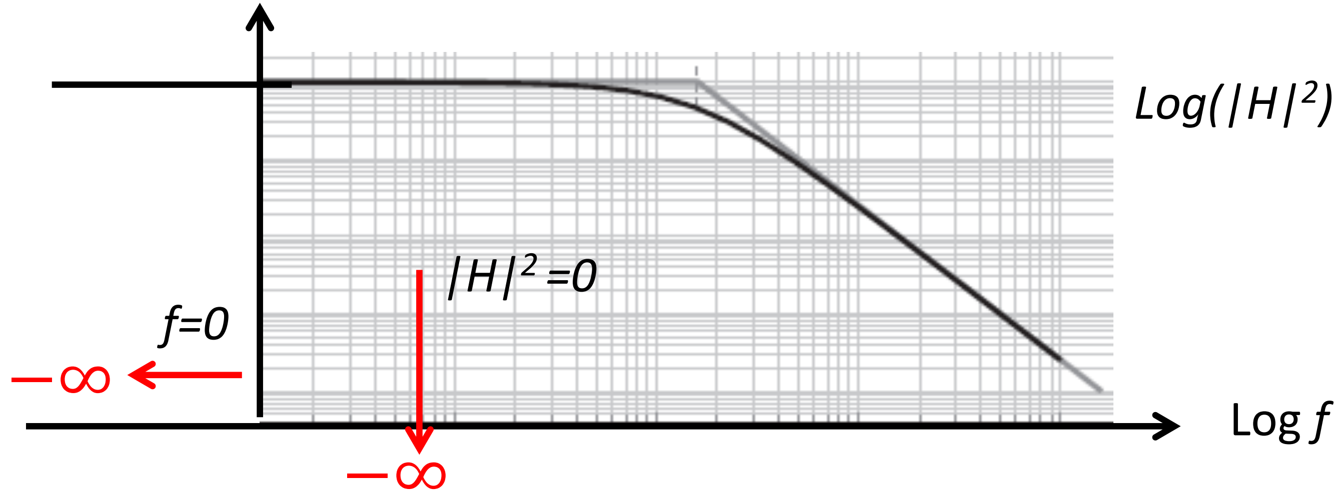
RC integrator: viewpoints on $|H(f)|^2$



LIN-LIN:
Linear vertical scale
Linear horizontal scale

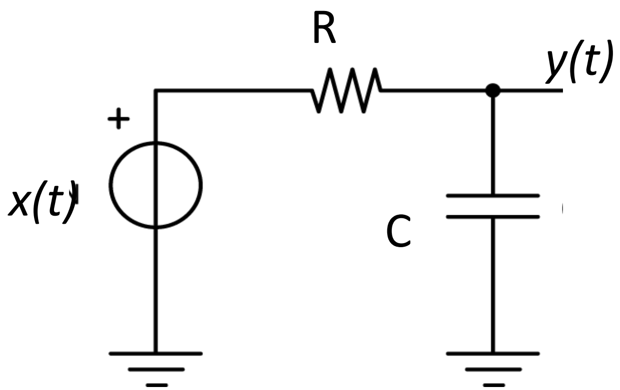


LIN-LOG:
Linear vertical scale
Logarithmic horizontal scale



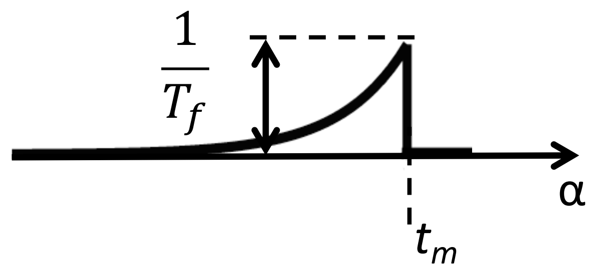
LOG-LOG (*Bode plot*):
Logarithmic vertical scale
Logarithmic horizontal scale

RC integrator (constant-parameter LPF)



Weighting function
in time

$$w_m(\alpha) = h(t_m - \alpha)$$

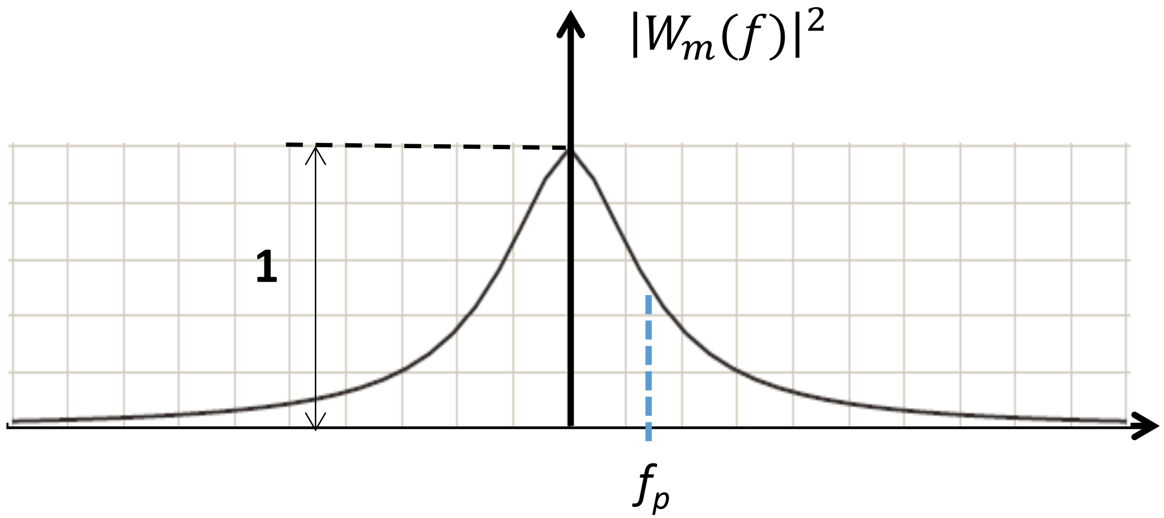


Output: can be seen as an **average over a time interval** ≈ $2T_f$ preceding t_m

Weighting function
in frequency

$$|W_m(f)|^2 = |H(f)|^2$$

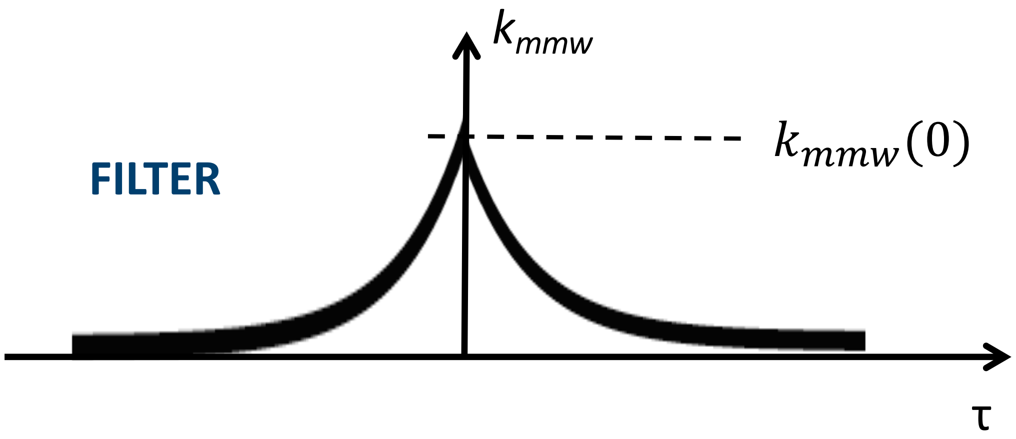
$$|W_m(f)|^2 = \frac{1}{1 + (2\pi f T_f)^2}$$



Output: can be seen as a **selection of the lower frequency components** up to ≈ f_p

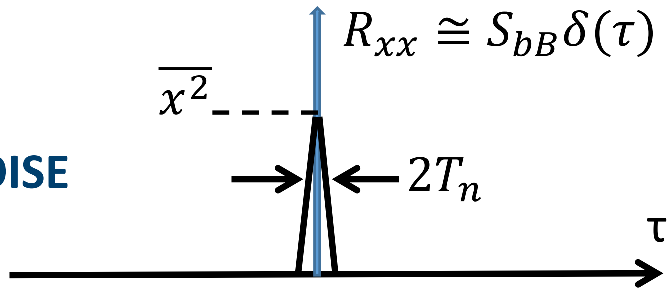
RC integrator: filtering wide-band noise:Time-domain analysis

FILTER



$$k_{mmw}(\tau) = \frac{1}{2T_f}\, e^{-\left|\frac{\tau}{T_f}\right|}$$

NOISE



$$\overline{y^2} = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot k_{mmw}(\tau)\, d\tau$$

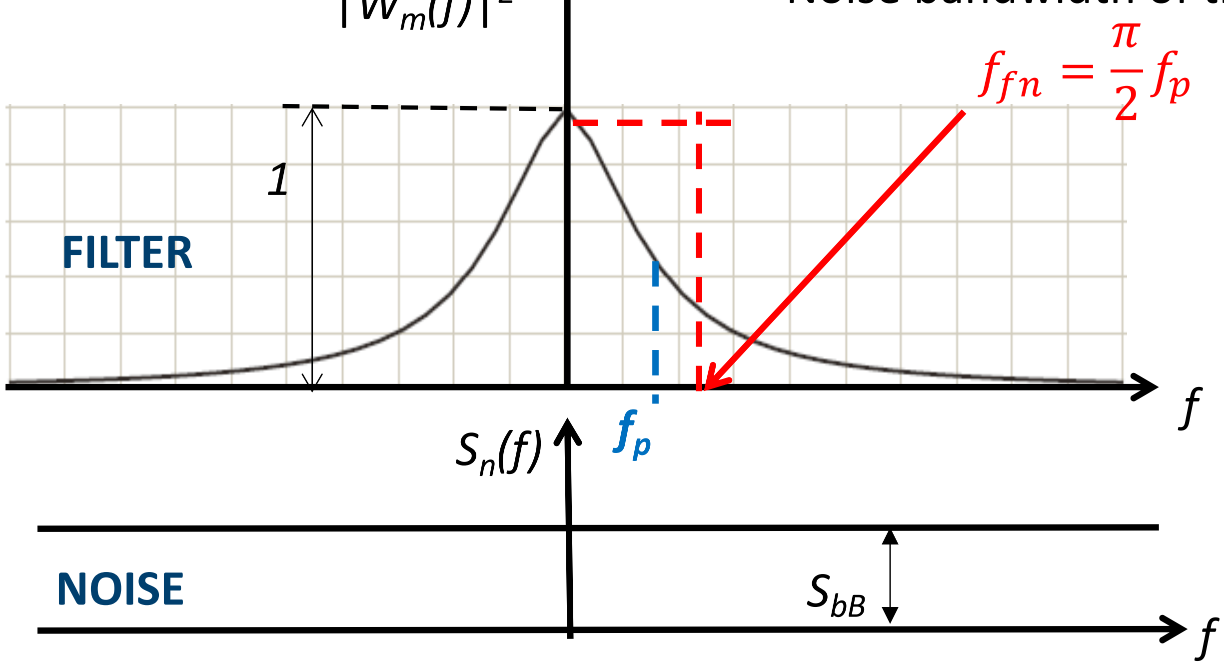
The noise is considered wide-band if it has autocorrelation much narrower than the filter weight autocorrelation, that is, if $T_n << T_f$

We can then approximate $R_{xx} \cong S_{bB}\delta(\tau)$ and obtain

$$\overline{\boldsymbol{y^2}} = \boldsymbol{S_{bB} \cdot k_{mmw}(0)} = \frac{\boldsymbol{S_{bB}}}{\boldsymbol{2T_f}}$$

RC integrator: filtering wide-band noise: Frequency-domain analysis

Noise bandwidth of the filter



$$|W_m(f)|^2 = |H(f)|^2 =$$
$$= \frac{1}{1 + (2\pi f T_f)^2}$$

$$\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_m(f)|^2 \, df$$

The noise is considered wide-band if it has spectrum much wider than the filter weighting spectrum, that is, if its bandlimit $f_n \gg f_p$

We can then approximate $S_x(f) \cong S_{bB}$ and obtain

$$\overline{y^2} = S_{bB} \cdot \int_{-\infty}^{\infty} |W_m(f)|^2 \, df = S_{bB} \cdot k_{mmw}(0) = \frac{S_{bB}}{2T_f} = S_{bB} \cdot 2 f_{fn}$$

RC integrator: noise band-width

Noise bandwidth f_{fn} of the filter:

defined with reference to a white noise input S_b as the bandwidth value to be employed for computing **simply by a multiplication** the output mean square noise

$$\overline{y^2} = S_{bB} \cdot 2 f_{fn}$$

Since it is

$$\overline{y^2} = S_{bB} \cdot \int_{-\infty}^{\infty} |W_m(f)|^2 \, df = S_{bB} \cdot k_{mmw}(0)$$

for any LPF the correct bandwidth limit f_{fn} is

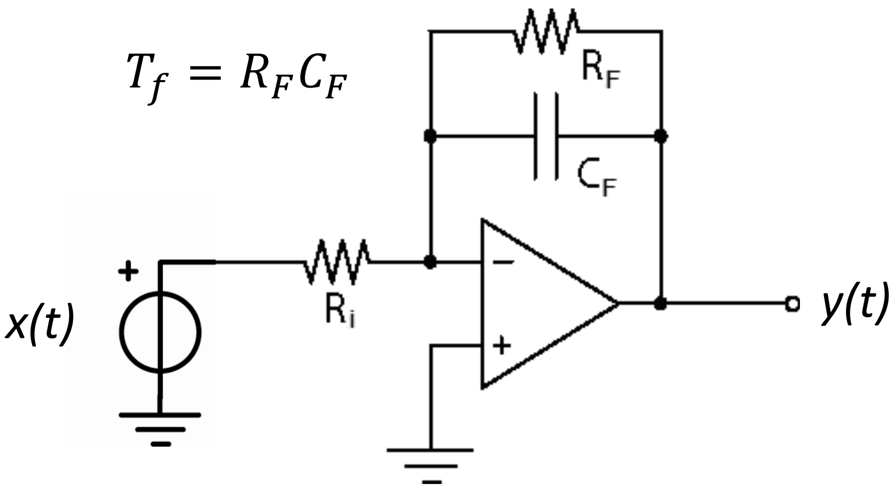
$$f_{fn} = \frac{k_{mmw}(0)}{2}$$

and in particular for the RC integrator

$$f_{fn} = \frac{1}{4T_f} = \frac{\pi}{2} f_p$$

RC integrator active filter

$$T_f = R_F C_F$$



TIME DOMAIN

$$-h(t) = \frac{R_F}{R_i}\frac{1}{T_f}1(t)e^{-t/T_f}$$

$$w_m(\alpha) = h(t_m - \alpha)$$

In comparison with the passive RC:

- still a constant-parameter filter
- same shape of the weighting
- dc gain = $\frac{R_F}{R_i}$ instead of 1

FREQUENCY DOMAIN

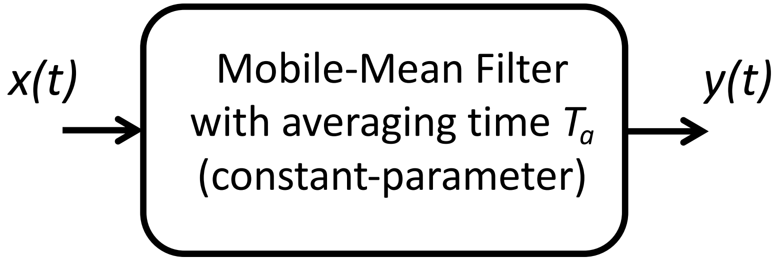
$$|H(f)|^2 = \left(\frac{R_F}{R_i}\right)^2 \frac{1}{1+(2\pi f T_f)^2}$$

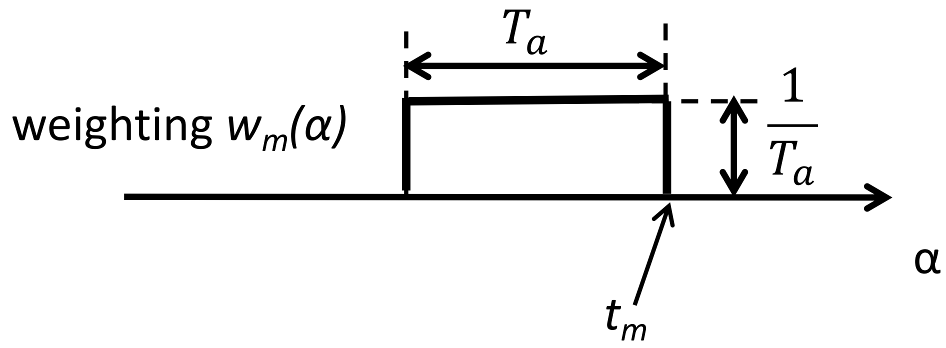
$$|W_m(f)|^2 = |H(f)|^2$$

dc gain $\quad |W_m(0)| = \frac{R_F}{R_i}$

Mobile-Mean Low-Pass Filter

Mobile-Mean Filter

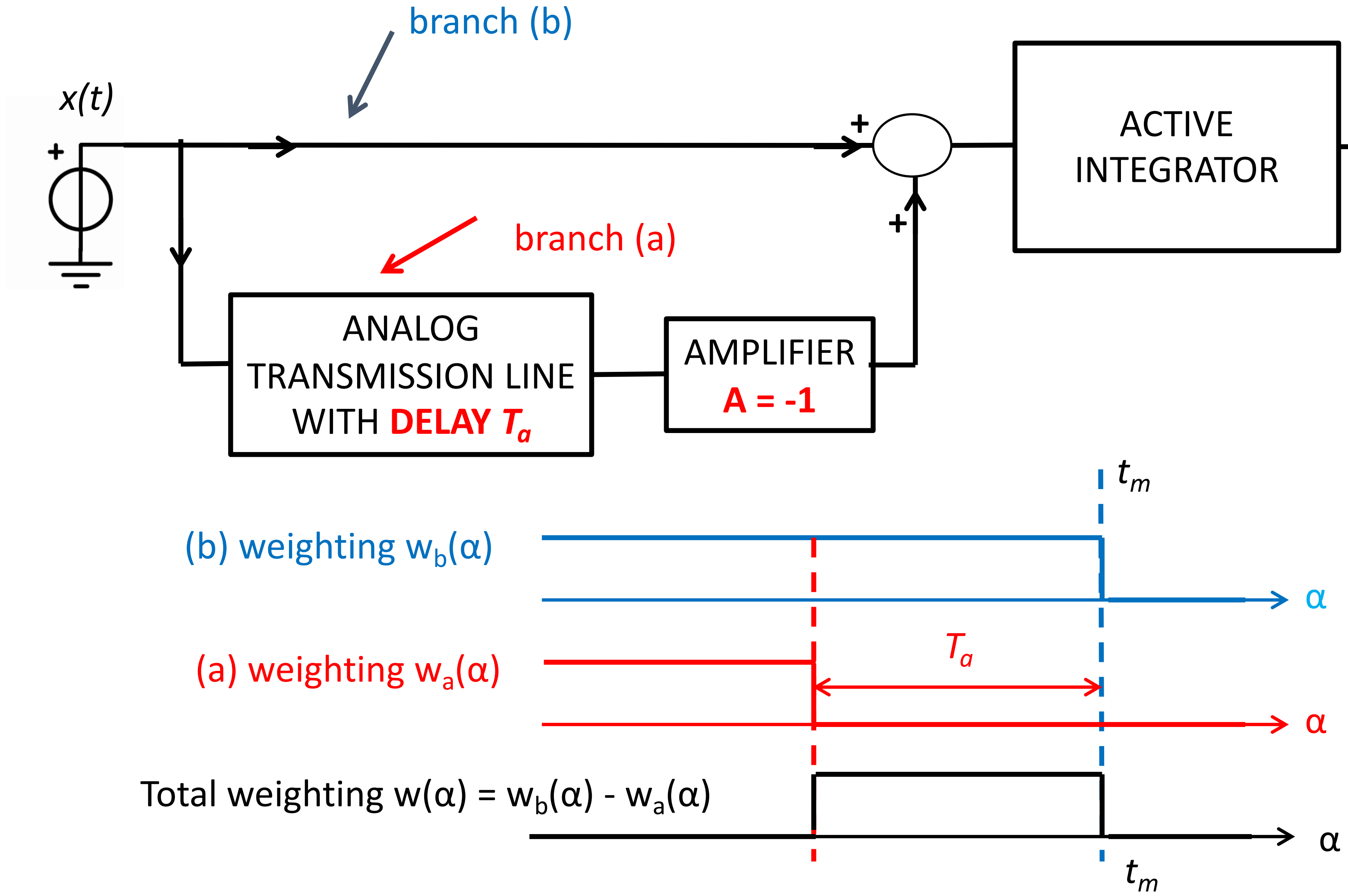




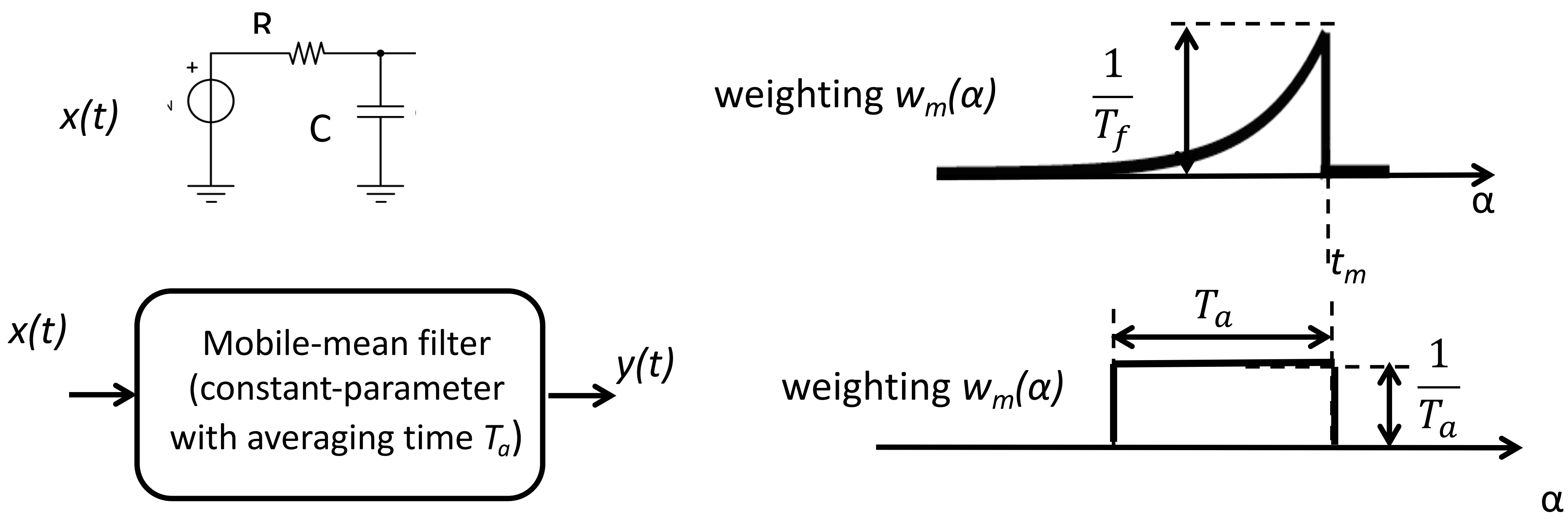
- **A mobile-mean filter (MMF)** produces at any time t_m an output *y*(t_m) which is not just the integral of the input *x(t)* over a time interval T_a that precedes t_m , but rather the **mean** value of the input *x(t)* **over the time interval** T_a , that is, the integral over T_a **divided by** T_a
- In order to obtain this, if we vary the averaging time T_a we must vary inversely the **weight** $1/T_a$ (this ensures constant area of $w_m(\alpha)$ i.e. constant DC gain).

The MMF is a **constant-parameter filter**: this is pointed out by the weighting function, which is the same for any readout time tm

The Mobile-Mean Filter is a constant-parameter filter



Mobile-mean filter versus RC-integrator



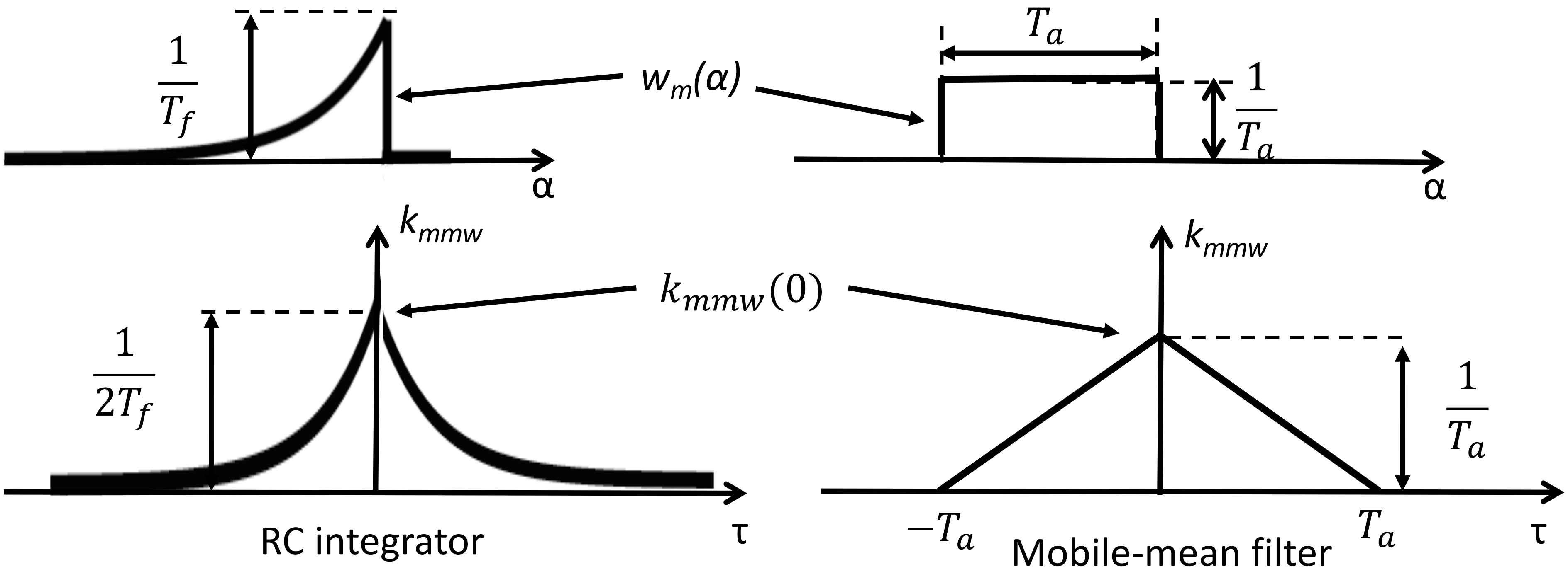
The mobile-mean filter produces an output $y(t_m)$ that is exactly the **mean value of the input x over the time interval T_a** preceding t_m .

When T_a is changed, the area of $w_m(\alpha)$ is kept constant, similarly to the case of the RC integrator when T_f is varied (the weight is reduced; the dc gain is kept constant)

Question: can we use a mobile-mean filter as equivalent to a given RC integrator for evaluating the result obtained by processing a signal with low-frequency content in presence of wide-band noise?

Answer: yes, the time T_a of the mobile-mean filter can be adjusted to produce equal output rms noise of the given RC integrator.

Mobile-mean filter equivalent to RC-integrator



Signal: the filters have **equal DC gain** (unity) and produce equal output with DC signal in.

Noise: for wide-band input noise the output noise is computed as

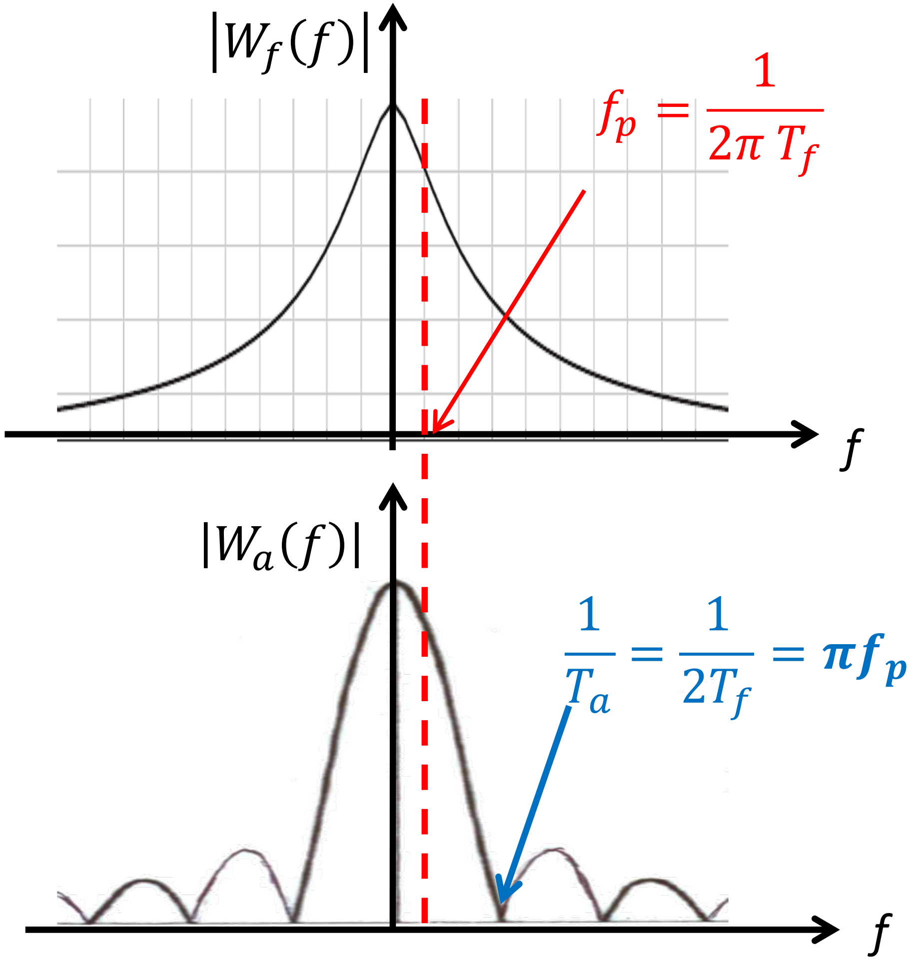
$$\overline{y^2} = S_{bB} \cdot k_{mmw}(0) = S_{bB} \cdot \int_{-\infty}^{\infty} {w_m}^2(\alpha)\, d\alpha$$

therefore, for having **equal output rms noise** it must be

$$\boxed{T_a = 2T_f}$$

Mobile-mean filter equivalent to RC-integrator

The weighting functions of the two filters in frequency domain plotted with the same scales clearly illustrate the equivalence



RC integrator
with $RC = T_f$

$$|W_f(f)| = |H_f(f)| = \frac{1}{\sqrt{1 + (2\pi f T_f)^2}}$$

Mobile-mean filter
with averaging interval $T_a = 2T_f$
(and unity gain)

$$|W_a(f)| = |H_a(f)| = \frac{|\sin 2\pi f T_f|}{2\pi f T_f}$$

Band-Width and Correlation Time of Low-Pass filters

«Rectangular» approximations of real filters

- The **noise bandwidth** f_{fn} of a low-pass filter is currently employed for evaluating the output noise of low-pass filters in **frequency-domain** computations.

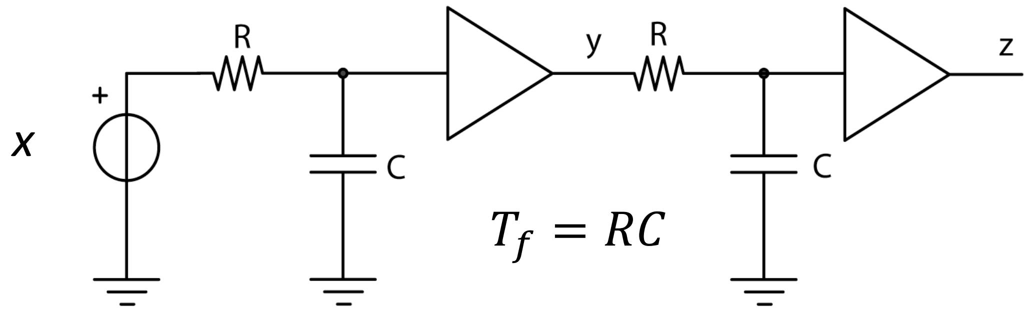
- **A REAL** filter that implements such a «rectangular weighting» in frequency **DOES NOT EXIST**: it would be a non-causal system, with δ-response that begins before the δ-pulse.

- **A REAL** filter that implements such a «rectangular weighting» in time **EXISTS:** it is the mobile-mean filter with averaging time $T_a = T_{fn}$.

- There are, however, practical limitations to the implementation of mobile-mean filters, mainly due to the impractical features and limited performance of the real analog transmission lines with long delay, namely delay longer than a few tens of nanoseconds.

Other constant-parameter LPF

For LPF filters with real poles, it is often easier to compute the noise bandwidth in time-domain rather than in frequency-domain, because it implies simple integrals (of exponentials and powers of *t*). **Example: cascade of two identical RC cells**



$$T_f = RC$$

$$h(t) = \frac{t}{{T_f}^2} e^{-\frac{t}{T_f}}$$

$$\overline{z^2} = S_{bB} \cdot k_{hh}(0) = S_{bB} \cdot \int_{-\infty}^{\infty} h(t)^2\, dt = S_{bB} \cdot \int_{-\infty}^{\infty} \left(\frac{t}{{T_f}^2}\right)^2 e^{-\frac{2t}{T_f}} dt$$

which integrated by parts gives

$$\overline{z^2} = S_{bB} \cdot \frac{1}{4T_f}$$

Since $\overline{z^2} = S_{bB}\, 2f_n$, the noise bandwidth f_n is

$$\boldsymbol{f_n = \frac{1}{8T_f}}$$