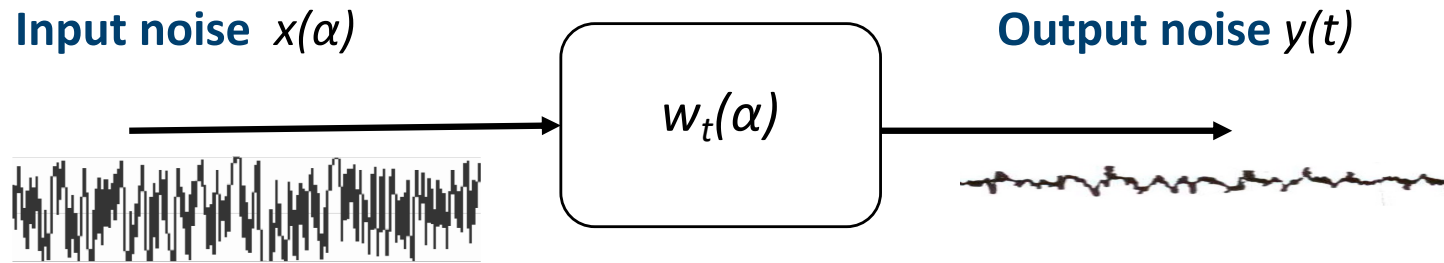


COURSE OUTLINE

- Introduction
- Signals and Noise
- **Filtering Noise**
- Sensors and associated electronics

- Mathematical Foundations
 - Filtering Stationary Noise
 - Filtering White Noise
 - Filtering Noise with Constant-Parameter Filters

Mathematical Foundations of Noise Processing by Linear Filters



characterized by

$$R_{xx}(\alpha, \alpha + \gamma) = \overline{x(\alpha)x(\alpha + \gamma)}$$

characterized by

$$R_{yy}(t, t + \tau) = \overline{y(t)y(t + \tau)}$$

The **output autocorrelation** can be obtained in terms of the **input autocorrelation** and of the filter **weighting** function :

$$\begin{aligned} R_{yy}(t_1, t_2) &= \overline{y(t_1)y(t_2)} = \\ &= \overline{\int_{-\infty}^{\infty} x(\alpha)w_1(\alpha)d\alpha \cdot \int_{-\infty}^{\infty} x(\beta)w_2(\beta)d\beta} = \iint_{-\infty}^{\infty} \overline{x(\alpha)x(\beta)} \cdot w_1(\alpha)w_2(\beta)d\alpha d\beta = \\ &= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta) w_1(\alpha)w_2(\beta)d\alpha d\beta \end{aligned}$$

by setting in evidence the intervals of autocorrelation at the input $\gamma = \beta - \alpha$ and at the output $\tau = t_2 - t_1$ can be expressed as

$$R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma) w_1(\alpha) w_2(\alpha + \gamma) d\alpha d\gamma$$

and in particular the **mean square noise** at time t_1 is

$$\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma) w_1(\alpha) w_1(\alpha + \gamma) d\alpha d\gamma$$

NB: these equations are **valid for all cases of noise and linear filtering**, that is, also for non-stationary input noise and for time-variant filters.

Filtering Stationary Noise

In case of stationary noise the input autocorrelation depends only on the time interval γ

$$R_{xx}(\alpha, \alpha + \gamma) = R_{xx}(\gamma)$$

The output autocorrelation is correspondingly simplified

$$\begin{aligned} R_{yy}(t_1, t_1 + \tau) &= \iint_{-\infty}^{\infty} R_{xx}(\gamma) w_1(\alpha) w_2(\alpha + \gamma) d\alpha d\gamma = \\ &= \int_{-\infty}^{\infty} R_{xx}(\gamma) \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha d\gamma \end{aligned}$$

NB: with stationary input noise:

- a) a constant parameter filter produces stationary output noise.
- b) a time-variant filter can produce a non-stationary output noise!

$$R_{yy}(t_1, t_1 + \tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha d\gamma$$

Denoting by $k_{12w}(\gamma)$ the **cross**correlation of the functions $w_1(\alpha)$ and $w_2(\alpha)$

$$k_{12w}(\gamma) = \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha$$

We can write

$$R_{yy}(t_1, t_1 + \tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{12w}(\gamma) d\gamma$$

For the **mean square noise** we must consider the **auto**correlation $k_{11w}(\alpha)$ of $w_1(\alpha)$

$$\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) d\gamma$$

With **stationary input noise** and for **any linear filter** (i.e. both constant-parameter and time variant filters) the **output noise** mean square value can be computed

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) d\gamma$$

By the Parseval theorem extension and recalling that:

$$F[k_{11w}(\gamma)] = |W_1(f)|^2$$

the output mean square noise can be computed also in the frequency domain

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2 df$$



Filtering White Noise

The fact that **White Stationary** noise has constant intensity (power)

$$R_{xx}(\alpha, \alpha + \gamma) = R_{xx}(\gamma) = S_b \cdot \delta(\gamma)$$

further simplifies the equation of the output autocorrelation

$$R_{yy}(t_1, t_1 + \tau) = S_b \int_{-\infty}^{\infty} w_1(\alpha)w_2(\alpha)d\alpha = S_b \cdot k_{12w}(\mathbf{0})$$

and of the output mean square value

$$\overline{y^2(t_1)} = S_b \cdot k_{11w}(\mathbf{0}) = S_b \int_{-\infty}^{\infty} w_1^2(\alpha)d\alpha$$

By Parseval theorem we have also

$$\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2 df$$



The **output mean square** of a filter that receives stationary noise can be computed

in the **time domain** as

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) d\gamma$$

in the **frequency domain** as

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2 df$$

and in case of **white** noise, i.e. with

$$R_{xx}(\gamma) = S_b \cdot \delta(\gamma)$$

$$S_x(f) = S_b$$



$$\overline{y^2(t_1)} = S_b \cdot k_{11w}(0) = S_b \int_{-\infty}^{\infty} w_1^2(\alpha) d\alpha$$



$$\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2 df$$

Filtering Noise with Constant-Parameter Filters

The constant-parameter filters:

- are completely characterized by the δ -response $h(t)$ in time and by the transfer function $H(f) = F[h(t)]$ in the frequency domain
- have weighting $w_m(\alpha)$ for acquisition at time t_m simply related to the δ -response

$$w_m(\alpha) = h(t - \alpha)$$

- therefore have

$$|W_m(f)|^2 = |H(f)|^2$$

They are **PERMUTABLE**. In a cascade of constant parameter filters, if the order of the various filters in the sequence is changed, the final output does NOT change.

They are **REVERSIBLE**. A constant parameter filter can change the shape of a signal, but it is always possible to find a restoring filter, that is, another constant parameter filter which restores the signal to the original shape.

The output autocorrelation is

$$\begin{aligned} R_{yy}(t_1, t_2) &= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta) w_1(\alpha) w_2(\beta) d\alpha d\beta = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta) \cdot h(t_1 - \alpha) h(t_2 - \beta) d\alpha d\beta = \\ &= \int_{-\infty}^{\infty} h(t_1 - \alpha) d\alpha \int_{-\infty}^{\infty} R_{xx}(\alpha, \beta) \cdot h(t_2 - \beta) d\beta = R_{xx}(\alpha, \beta) * h(\beta) * h(\alpha) \end{aligned}$$

and taking into account that:

- the **stationary** input autocorrelation depends only on the interval $\gamma = \beta - \alpha$
- $d\beta = d\gamma$
- $d\alpha = -d\gamma$
- the output autocorrelation is also **stationary** and depends only on the interval τ

$$R_{yy}(\tau) = R_{xx}(\gamma) * h(\gamma) * h(-\gamma) = R_{xx}(\gamma) * k_{hh}(\gamma)$$

and therefore

$$S_y(f) = S_x(f) \cdot |H(f)|^2$$



CONSTANT-PARAMETER filters with Stationary input noise

From the output autocorrelation $R_{yy}(\tau) = R_{xx}(\gamma) * k_{hh}(\gamma)$ we obtain for the **output mean square value**:

$$\overline{y^2} = R_{yy}(0) = \int_{-\infty}^{\infty} R_{xx}(\gamma) k_{hh}(\gamma) d\gamma$$

and by Parseval's theorem

$$\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df$$

In the case of **white** input noise $R_{xx}(\gamma) = S_b \delta(\gamma)$ and therefore

$$\overline{y^2} = S_b k_{hh}(0)$$

$$\overline{y^2} = S_b \int_{-\infty}^{\infty} |H(f)|^2 df$$

