COURSE OUTLINE

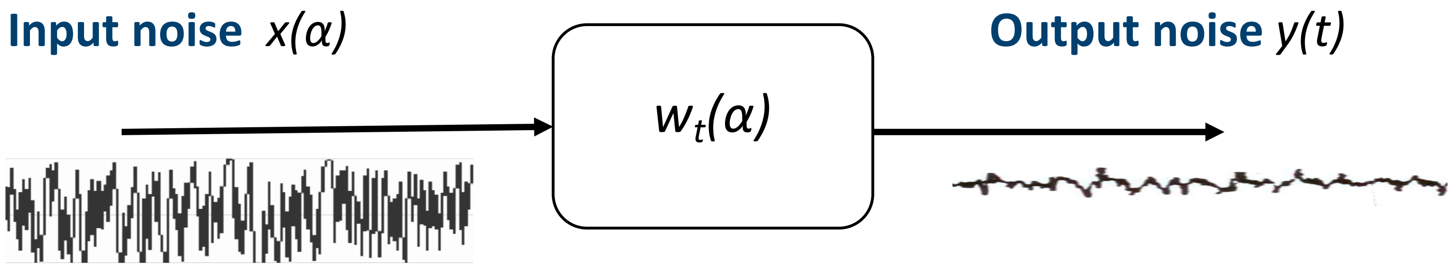
- Introduction
- Signals and Noise
- **Filtering Noise**
- Sensors and associated electronics

Processing Noise with Linear Filters

- Mathematical Foundations
  - Filtering Stationary Noise
    - Filtering White Noise
    - Filtering Noise with Constant-Parameter Filters

Mathematical Foundations of Noise Processing by Linear Filters

Noise filtering



characterized by

$$R_{xx}(\alpha, \alpha+\gamma) = \overline{x(\alpha)x(\alpha+\gamma)}$$

characterized by

$$R_{yy}(t, t+\tau) = \overline{y(t)y(t+\tau)}$$

The **output autocorrelation** can be obtained in terms of the **input autocorrelation** and of the filter **weighting** function :

$$R_{yy}(t_1, t_2) = \overline{y(t_1)y(t_2)} =$$

$$= \overline{\int_{-\infty}^{\infty} x(\alpha)w_1(\alpha)d\alpha \cdot \int_{-\infty}^{\infty} x(\beta)w_2(\beta)d\beta} = \iint_{-\infty}^{\infty} \overline{x(\alpha)x(\beta)} \cdot w_1(\alpha)w_2(\beta)d\alpha\, d\beta =$$

$$= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta)\, w_1(\alpha)w_2(\beta)d\alpha\, d\beta$$

Noise filtering

by setting in evidence the intervals of autocorrelation at the input $\gamma = \beta - \alpha$ and at the output $\tau = t_2 - t_1$ can be expressed as

$$\boxed{R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma}$$

and in particular the **mean square noise** at time t_1 is

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha)\, w_1(\alpha + \gamma) d\alpha\, d\gamma}$$

NB: these equations are **valid for all cases of noise and linear filtering**, that is, also for non-stationary input noise and for time-variant filters.

Filtering Stationary Noise

Filering Stationary Noise

In case of stationary noise the input autocorrelation depends only on the time interval γ

$$R_{xx}(\alpha, \alpha + \gamma) = R_{xx}(\gamma)$$

The output autocorrelation is correspondingly simplified

$$R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma =$$

$$= \int_{-\infty}^{\infty} R_{xx}(\gamma) \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma$$

NB: with stationary input noise:

a) a constant parameter filter produces stationary output noise.

b) a time-variant filter can produce a non-stationary output noise!

Filtering Stationary Noise

$$R_{yy}(t_1, t_1 + \tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha \, d\gamma$$

Denoting by $k_{12w}(\gamma)$ the **cross**correlation of the functions $w_1(\alpha)$ and $w_2(\alpha)$

$$k_{12w}(\gamma) = \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha$$

We can write

$$\boxed{R_{yy}(t_1, t_1 + \tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{12w}(\gamma) \, d\gamma}$$

For the **mean square noise** we must consider the **auto**correlation $k_{11w}(\alpha)$ of $w_1(\alpha)$

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) \, d\gamma}$$

Filtering Stationary Noise

With **stationary input noise** and for **any linear filter** (i.e. both constant-parameter and time variant filters) the **output noise** mean square value can be computed

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma}$$

By the Parseval theorem extension and recalling that:

$$F[k_{11w}(\gamma)] = |W_1(f)|^2$$

the output mean square noise can be computed also in the frequency domain

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2\, df}$$

Filtering White Noise

Filtering White Stationary noise

The fact that White Stationary noise has constant intensity (power)

$$R_{xx}(\alpha, \alpha+\gamma) = R_{xx}(\gamma) = S_b \cdot \delta(\gamma)$$

further simplifies the equation of the output autocorrelation

$$\boxed{R_{yy}(t_1, t_1+\tau) = S_b \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha) d\alpha = \boldsymbol{S_b \cdot k_{12w}\,(0)}}$$

and of the output mean square value

$$\boxed{\overline{y^2(t_1)} = S_b \cdot k_{11w}\,(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha}$$

By Parseval theorem we have also

$$\boxed{\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2\, df}$$

Filtering Stationary Noise

The **output mean square** of a filter that receives stationary noise can be computed

in the **time domain** as

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma$$

in the **frequency domain** as

$$\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2\, df$$

and in case of **white** noise, i.e. with

$$R_{xx}(\gamma) = S_b \cdot \delta(\gamma) \qquad\qquad S_x(f) = S_b$$

$$\overline{y^2(t_1)} = S_b \cdot k_{11w}(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha$$

$$\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2\, df$$

Filtering Noise with Constant-Parameter Filters

About CONSTANT-PARAMETER filters

The constant-parameter filters:

- are completely characterized by the δ-response *h(t)* in time and by the transfer function *H(f) = F[h(t)]* in the frequency domain

- have weighting $w_m(\alpha)$ for acquisition at time t_m simply related to the δ-response

$$\boldsymbol{w_m(\alpha) = h(t - \alpha)}$$

- therefore have

$$|\boldsymbol{W_m(f)}|^2 = |\boldsymbol{H(f)}|^2$$

They are **PERMUTABLE**. In a cascade of constant parameter filters, if the order of the various filters in the sequence is changed, the final output does NOT change.

They are **REVERSIBLE**. A constant parameter filter can change the shape of a signal, but it is always possible to find a restoring filter, that is, another constant parameter filter which restores the signal to the original shape.

CONSTANT-PARAMETER filters with stationary input noise

The output autocorrelation is

$$R_{yy}(t_1,t_2) = \iint_{-\infty}^{\infty} R_{xx}(\alpha,\beta)\, w_1(\alpha) w_2(\beta) d\alpha\, d\beta = \iint_{-\infty}^{\infty} R_{xx}(\alpha,\beta) \cdot h(t_1 - \alpha) h(t_2 - \beta) d\alpha\, d\beta =$$

$$= \int_{-\infty}^{\infty} h(t_1 - \alpha) d\alpha \int_{-\infty}^{\infty} R_{xx}(\alpha,\beta) \cdot h(t_2 - \beta) d\beta = R_{xx}(\alpha,\beta) * h(\beta) * h(\alpha)$$

and taking into account that:

- the stationary input autocorrelation depends only on the interval $\gamma = \beta - \alpha$
- $d\beta = d\gamma$
- $d\alpha = -\, d\gamma$
- the output autocorrelation is also stationary and depends only on the interval τ

$$\boxed{R_{yy}(\tau) = R_{xx}(\gamma) * h(\gamma) * h(-\gamma) = R_{xx}(\gamma) * k_{hh}(\gamma)}$$

and therefore

$$\boxed{S_y(f) = S_x(f) \cdot |H(f)|^2}$$

CONSTANT-PARAMETER filters with Stationary input noise

From the output autocorrelation $R_{yy}(\tau) = R_{xx}(\gamma) * k_{hh}(\gamma)$ we obtain for the **output mean square value:**

$$\boxed{\overline{y^2} = R_{yy}(0) = \int_{-\infty}^{\infty} R_{xx}(\gamma)\, k_{hh}(\gamma) d\gamma}$$

and by Parseval's theorem

$$\boxed{\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df}$$

In the case of **white** input noise $R_{xx}(\gamma) = S_b \delta(\gamma)$ and therefore

$$\boxed{\overline{y^2} = S_b\, k_{hh}(0)}$$

$$\boxed{\overline{y^2} = S_b \int_{-\infty}^{\infty} |H(f)|^2 df}$$