COURSE OUTLINE

- Introduction
- Signals and **Noise: 3) Analysis and Simulation**
- Filtering
- Sensors and associated electronics

Noise Analysis and Simulation

- White Noise
- Band-Limited White Noise (Wide-Band Noise)
- Basic Parameters of Wide-Band Noise
- Foundations of White-Noise Filtering

White Noise

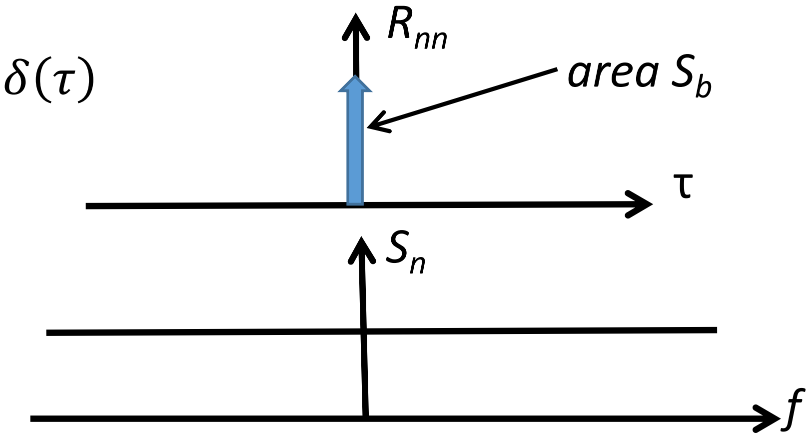
White Noise (stationary)

IDEAL «white» noise
is a concept extrapolated from Johnson noise and shot noise
defined by its **essential** feature:
no autocorrelation at any time distance τ, no matter how small

$$R_{nn}(\tau) = S_b \cdot \delta(\tau)$$

and therefore constant spectrum

$$S_n(f) = S_b$$



In reality such a noise does not exist: it would have divergent power $\overline{n^2} \rightarrow \infty$

REAL «white» noise has

- **Very small width** of autocorrelation, shorter than the minimum time interval of interest in the actual case and therefore approximated to zero
- **Very wide band** with constant spectral density S_b , wider than the maximum frequency of interest in the actual case and therefore approximated to infinite

White Noise (non-stationary)

Also in non-stationary cases the IDEAL «white» noise
is defined by the **essential** characteristic feature:

no correlation at any finite time distance τ, no matter how small,

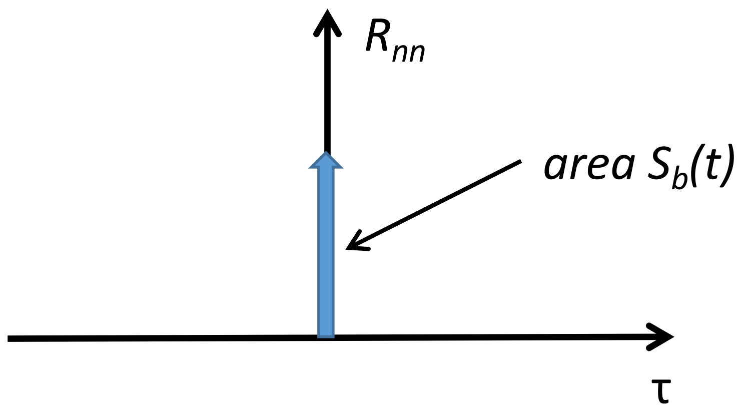
but the noise intensity is no more constant, it varies with time *t*

that is

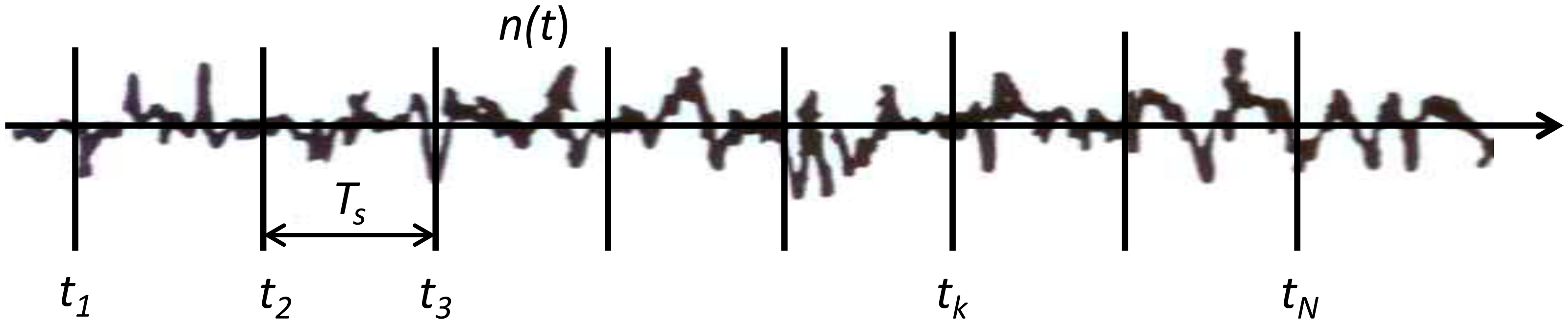
the autocorrelation function is δ-like,

but has **time-dependent area $S_b(t)$**

$$R_{nn}(t, t+\tau) = S_b(t) \cdot \delta(\tau)$$



Filtering white noise is simple



For clarity, let's consider a discrete case:

linear filtering in digital signal processing:

- Sample n_1 at t_1 and multiply by a weight w_1 ,
- sample n_2 at $t_2 = t_1 + T_s$ and multiply by a weight w_2 and sum
- and so on ....

The filtered noise n_f is

$$n_f = w_1 n_1 + w_2 n_2 + \; \ldots\ldots = \sum_{k=1}^{N} w_k n_k$$

and its mean square value is

$$\overline{n_f{}^2} = \overline{w_1{}^2 n_1{}^2 + w_2{}^2 n_2{}^2 + \; \ldots + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1{}^2 \overline{n_1{}^2} + w_2{}^2 \overline{n_2{}^2} + \; \ldots \; w_1 \, w_2 \overline{n_1 n_2} + w_1 w_3 \overline{n_1 n_3} + \cdots$$

Filtering white noise is simple

$$\overline{n_f{}^2} = \overline{w_1{}^2 n_1{}^2 + w_2{}^2 n_2{}^2 + \ \ldots + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1{}^2 \overline{n_1{}^2} + w_2{}^2 \overline{n_2{}^2} + \ \ldots\ w_1\, w_2 \overline{n_1 n_2} + w_1 w_3 \overline{n_1 n_3} + \ldots$$

If noise at interval T_s is **not correlated**, then all rectangular terms vanish

$$\overline{n_1 n_2} = \overline{n_1 n_2} = \cdots = 0$$

and the result is **simply a sum of squares**, even in case of non-stationary noise

$$\overline{n_f{}^2} = w_1{}^2 \overline{n_1{}^2} + w_2{}^2 \overline{n_2{}^2} + \ \ldots =$$

$$= \sum_{k=1}^{N} w_k{}^2 \overline{n_k{}^2}$$

If the noise is stationary

$$\overline{n_1{}^2} = \overline{n_2{}^2} = \overline{n_3{}^2} = \cdots = \overline{n^2}$$

there is a further simplification

$$\overline{n_f{}^2} = \overline{n^2}\ (w_1{}^2 + w_2{}^2 + \ \ldots) =$$

$$= \overline{n^2}\ \sum_{k=1}^{N} w_k{}^2$$

we will see later that also with continuous filtering white noise brings similar simplification

Band-Limited White Noise

or

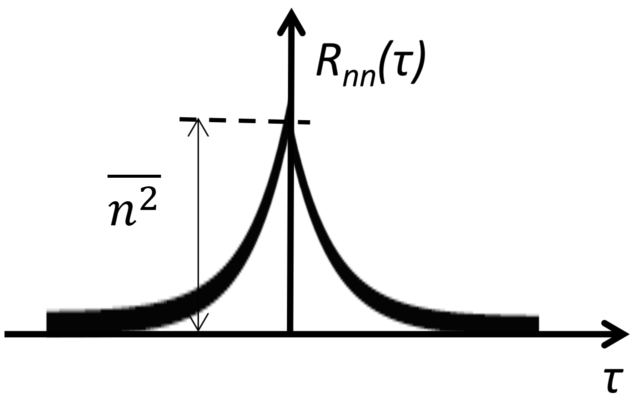
Wide-Band Noise

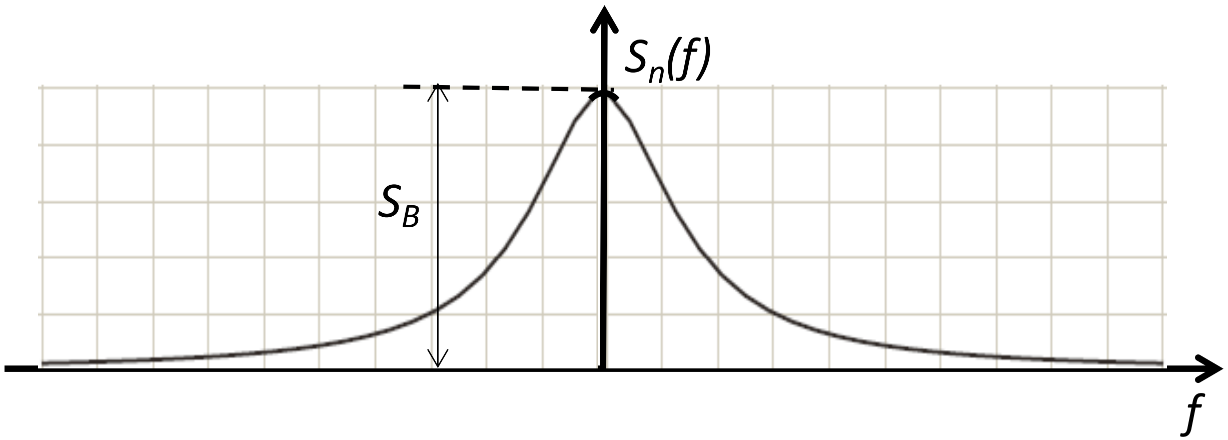
Band-limited white noise (wide-band noise)

- **Real white noise** = white noise with band limited at high frequency
- *The limit may be inherent in the noise source or due to low-pass filtering enforced by the circuitry*. **Anyway, in all real cases there is such a limit**
- A frequent typical case is the **Lorentzian** spectrum:
  band limited by a **simple pole** with time constant T_p , pole frequency $f_p = 1/ 2\pi T_p$

$$R_{nn}(\tau) = \overline{n^2} e^{-\frac{|\tau|}{T_p}}$$

$$S_n(f) = \frac{S_B}{1 + \left(2\pi f T_p\right)^2}$$



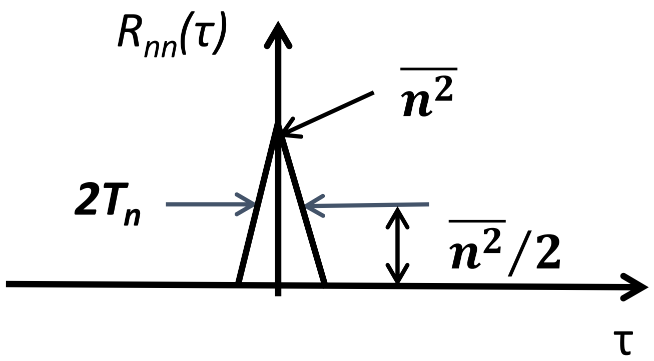


Basic Parameters of Wide-Band Noise

Simplified description of wide-band noise

The true $R_{nn}(\tau)$ and S_n *(f)* can be **approximated retaining the noise main features**:

a) equal **mean square $\overline{n^2}$** and b) equal spectral **density S_b**



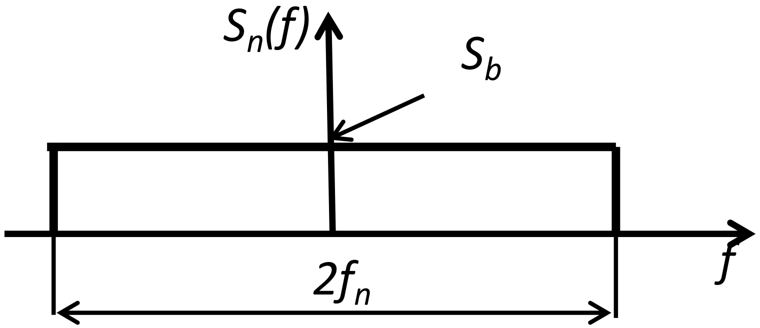
in time: $R_{nn}(\tau)$ triangular approx, half-width $2T_n$

a) **equal msq noise** : $R_{nn}(0) = \overline{n^2}$

b) **equal spectral density**: [area of $R_{nn}(\tau)$] = S_b ,

(i.e. $\overline{n^2}\, 2\, T_n = S_b$)

Correlation width = $\Delta\tau = 2T_n$



in frequency: S_n *(f)* rectang approx, half-width f_n

a) **equal msq noise** : [area of S_n *(f)*] = $\overline{n^2}$

(i.e. $S_b 2\, f_n = \overline{n^2}$)

b) **equal spectral density**: $S_n(0) = S_b$

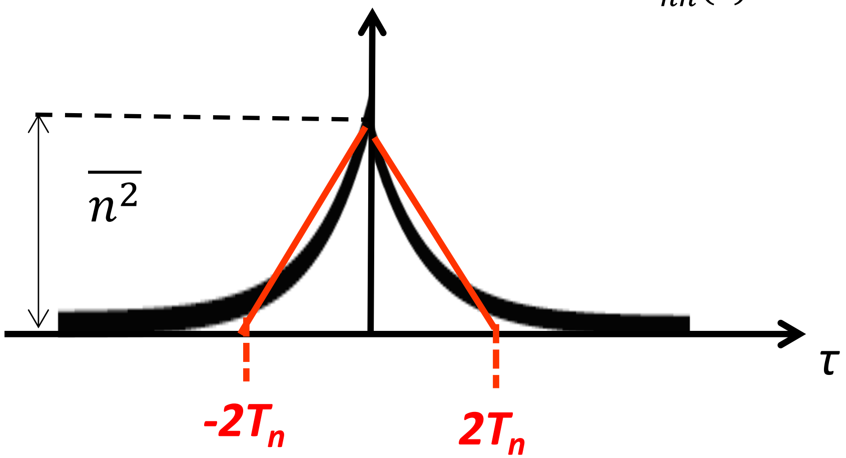
Noise bandwidth: $\Delta f = 2f_n$

Note that $\Delta\tau \cdot \Delta f = 1$ which is consistent with $S_n(f) = F[R_{nn}(\tau)]$

Simplified description of Lorentzian spectrum

Time

$$R_{nn}(\tau) = \overline{n^2} e^{-\frac{|\tau|}{T_p}}$$

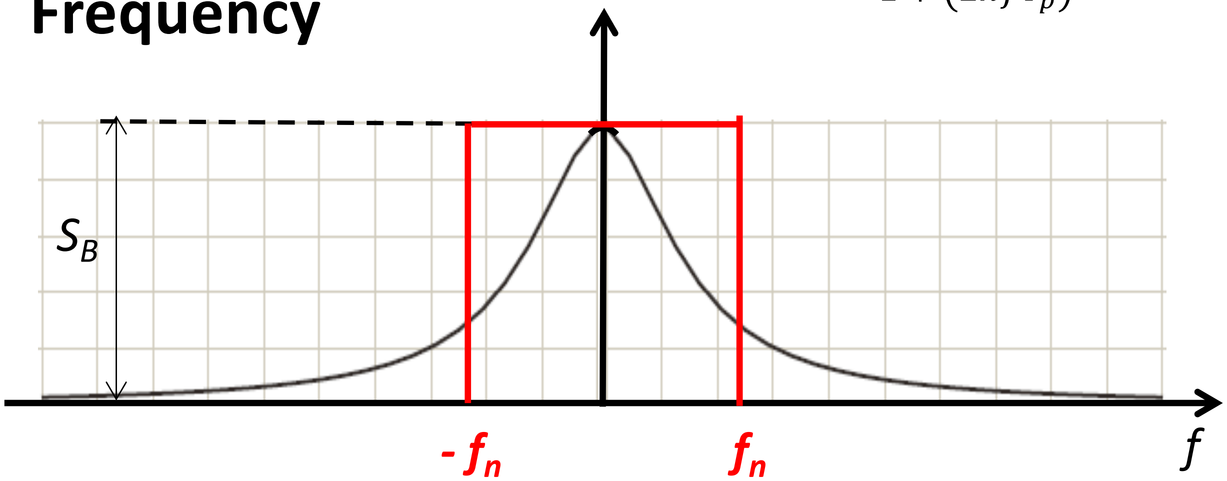


$$S_B = \int_{-\infty}^{\infty} R_{nn}(\tau)\ d\tau = \overline{n^2}\ 2T_p$$

$$\boxed{T_n = T_p}$$

Frequency

$$S_n(f) = \frac{S_B}{1 + (2\pi f T_p)^2}$$



$$\overline{n^2} = \int_{-\infty}^{\infty} S_n(f)\ df = S_B \frac{1}{2T_p}$$

$$\boxed{2f_n = \frac{1}{2T_p}}$$

Note that $f_n \neq f_p$, namely $f_n = \dfrac{1}{4T_p} = \dfrac{\pi}{2} f_p$

Foundations of White-Noise Filtering

Noise filtering clarified by the Poisson pulse model

Noise is a random superposition of **elementary pulses**

↓

The elementary pulse type (i.e. pulse waveform and its F-transform) defines the noise type (i.e. autocorrelation shape and spectrum shape)

The passage through a linear constant-parameter filter modifies the elementary pulse type

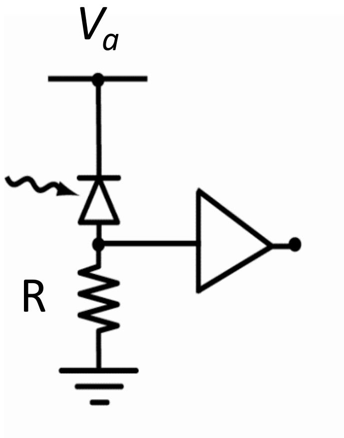
↓

The pulse modification causes a corresponding modification of the noise

↓

Noise filtering can thus be understood, studied and evaluated by understanding, studying and evaluating the filtering of the elementary pulses

Low-pass filtering of White noise



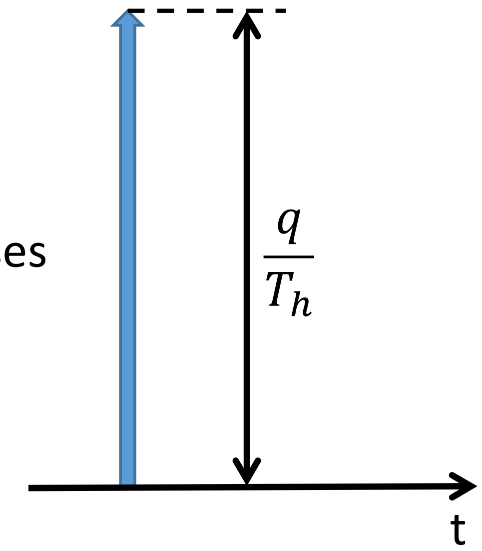
Shot current white noise:

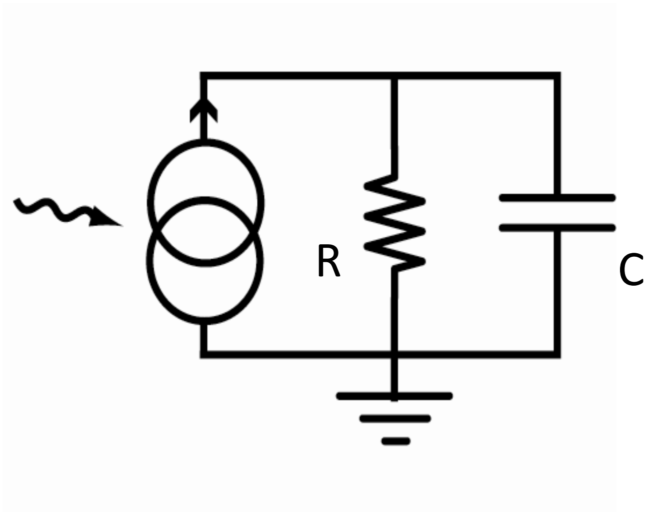
$S_i(f) \cong qI$

$R_{ii}(\tau) \cong qI\delta(\tau)$

Diode current: elementary short pulses with rate $p = I/q$, $T_h \cong 100ps$

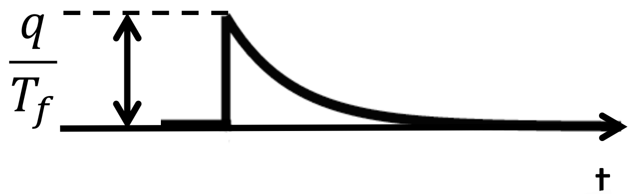
$$\boldsymbol{qh(t) \cong q\delta(t)}$$





Current in R: elementary exponential pulses with rate $p = I/q$, $T_f = RC$

$$\boldsymbol{q \cdot f(t)}$$



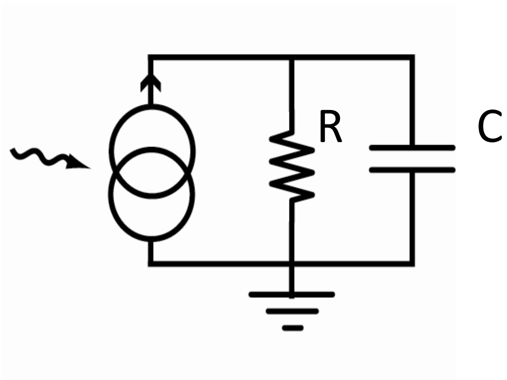
$f(t) = \frac{1}{T_f} 1(t) e^{-t/T_f}$ $\qquad$ $F(f) = \frac{1}{1+j2\pi f T_f}$

e.g. with $R = 100k\Omega$ and $C = 10pF$ we have $T_f = 1\mu s$

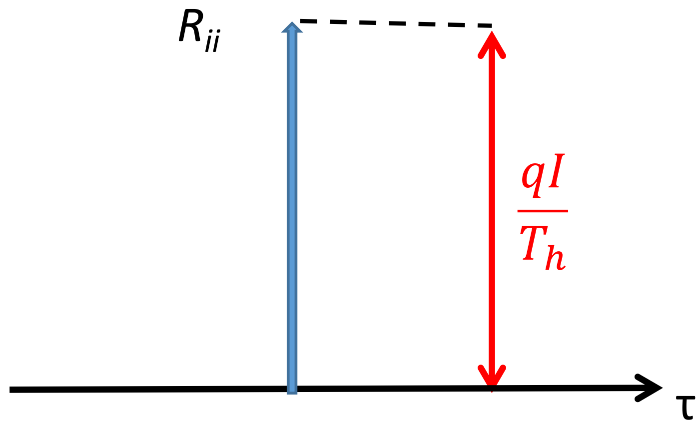
Low-pass filtering of White noise: time domain view

Input noise (*current in the diode*):
δ-like autocorrelation (width $\approx 100ps$)

$$R_{ii}(\tau) \cong qI\delta(\tau) \quad \text{for } \tau >> 100ps$$



$T_f = R\,C = 1\mu s$

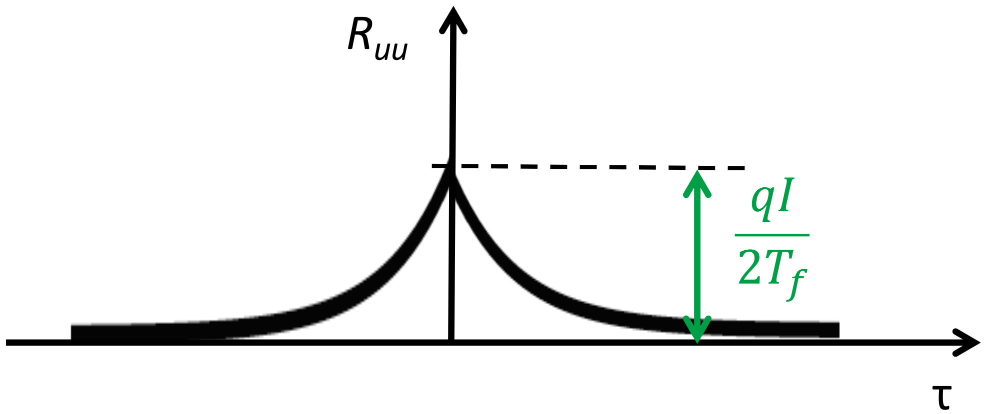


***To compare msq** values of noise **before** and **after** filtering compare the **central values** of autocorrelation functions*

Output noise (*current in R*):
autocorrelation function

$$R_{uu}(\tau) = qI \cdot k_{ff}(\tau)$$

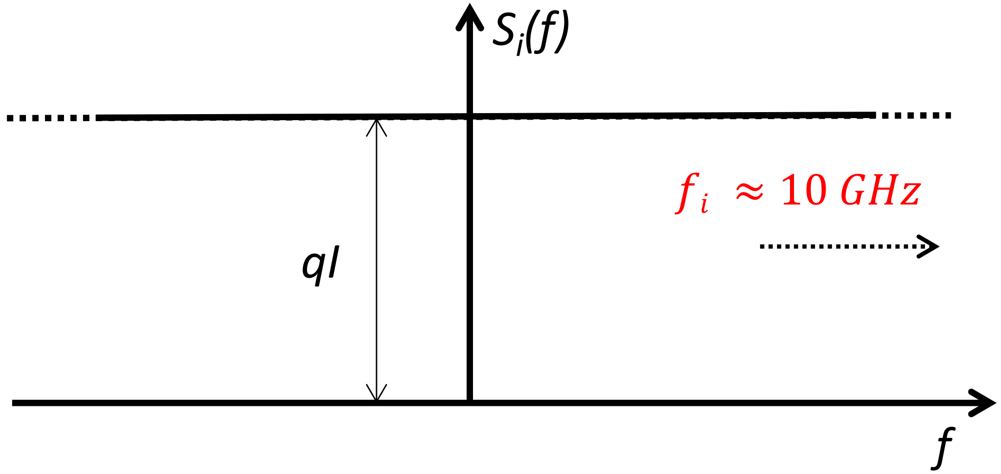
$$k_{ff}(\tau) = \frac{1}{2T_f}\, e^{-\left|\frac{\tau}{T_f}\right|}$$

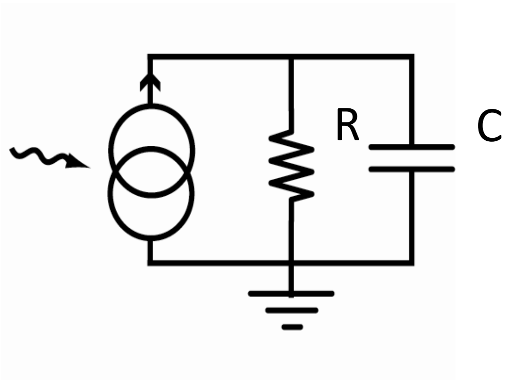


Low-pass filtering of White noise: frequency domain view

Input noise (*diode current*):
spectral density S_i constant
(bandwidth $f_i \approx 10\ GHz$)

$$S_i(f) = S_b \quad \text{for } f << 10\ GHz$$





$T_f = R\,C = 1\mu s$

To compare *msq values of noise* ***before*** *and* ***after*** *filtering compare the* ***areas*** *of input and output spectral densities*

Output noise (*current in R*):
spectral density function $S_u(f)$

$$S_u(f) = qI \cdot |F(f)|^2$$

$$|F(f)|^2 = \frac{1}{1 + (2\pi f T_f)^2}$$