



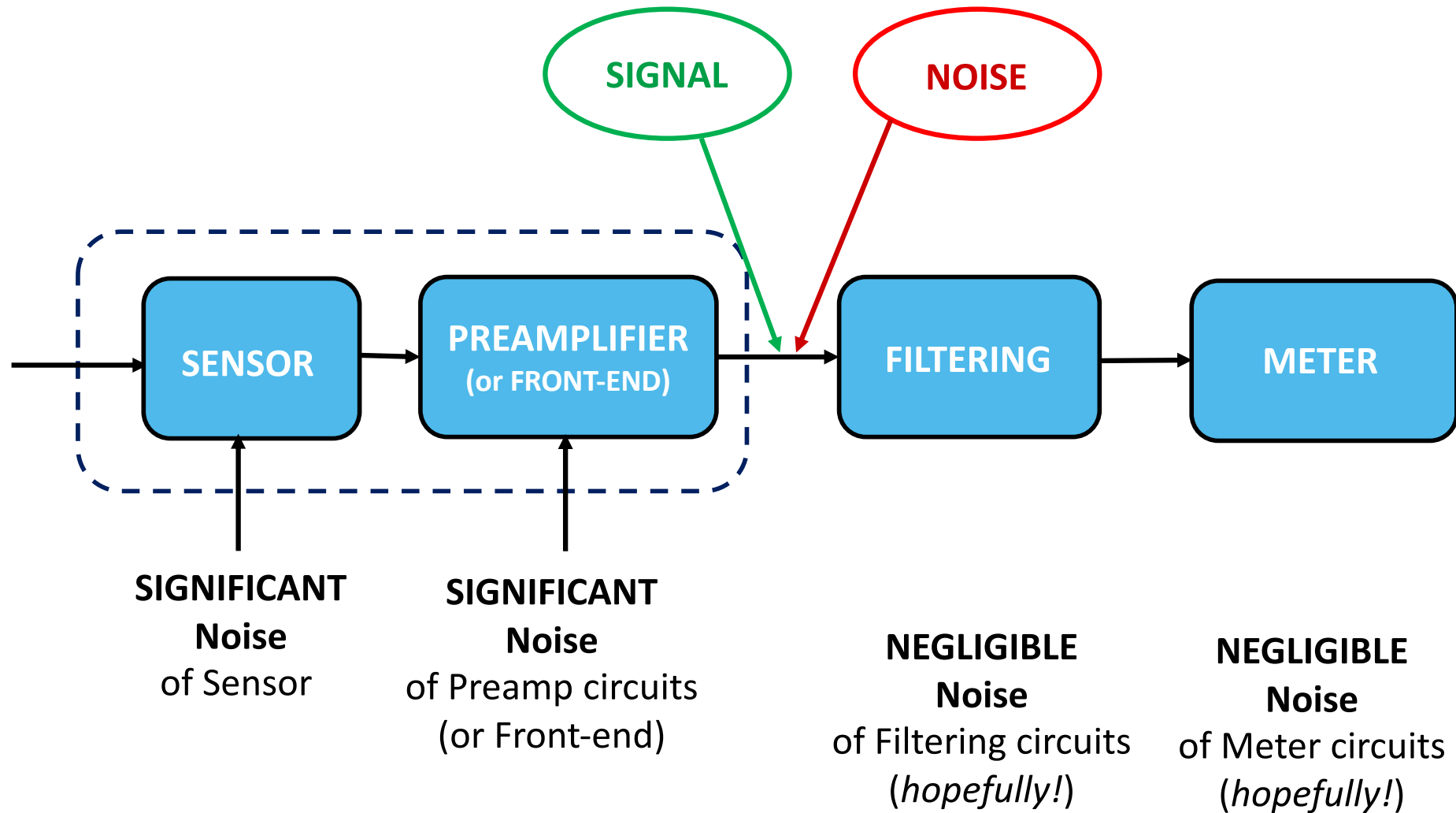
COURSE OUTLINE

- Introduction
- **Signals** and Noise
- Filtering
- Sensors and associated electronics



Set-Up for Sensor Measurements

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- Time domain and frequency domain analysis
- Note on truncated signals: example of how we play with signals
- Energy signals and correlation functions
- Energy Spectrum
- Power signals, Correlation Functions and Power Spectrum

- *Book: Fourier transform and properties*
- *Book: Crosscorrelation and Convolution*





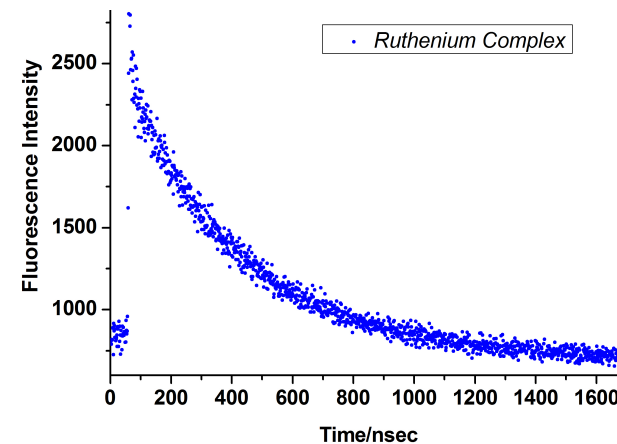
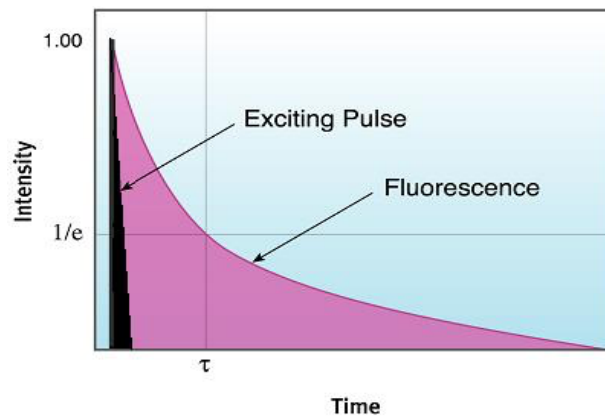
Time domain and frequency domain analysis of signals

- **Signals** = electric variables x (voltage, current ...) that carry information
- In the domain of time t : **deterministic** functions $x = x(t)$



Example: exponential pulse

$$x = 1(t)e^{-t/T}$$



In the domain of frequency f (Fourier transform domain) can be considered linear superposition (sum) of elementary sinusoid components

1. Segnali e sistemi continui

Segnali continui: scalino, impulso, esponenziali complessi, operazioni elementari sui segnali.

Sistemi Lineari Tempo-Invarianti: risposta impulsiva, convoluzione, correlazione.

Rappresentazione dei segnali nel dominio della frequenza: trasformata e serie di Fourier.

Densità spettrale di energia e potenza.

Dal tempo continuo al tempo discreto: teorema del campionamento, ricostruzione ed equivocazione nel tempo e nelle frequenze.

Trasformata di Fourier di Segnali discreti, energia, DFT.

2. Probabilità, processi casuali

Introduzione: definizioni, variabili casuali discrete e continue. Distribuzione e densità.

Probabilità condizionate: statistica indipendenza, regola di Bayes.

Prove ripetute (Bernoulli) e teoremi limite. Valori medi, quantili, momenti e correlazione.

Distribuzioni notevoli: normale, uniforme, binomiale, Poisson, esponenziale.

3. Processi casuali

Processi casuali: realizzazioni e medie d'insieme. Stazionarietà ed ergodicità.

Autocorrelazione e spettro di potenza. Predizione. Processi attraverso sistemi lineari.

Esempi e applicazioni: Rumore bianco e di quantizzazione.

Densità spettrale di processi modulati in fase e processi ciclostazionari. Stima spettrale non parametrica: periodogramma.

4. Informazione e trasmissione

Codifica di sorgente: quantizzazione, codifica di Huffman e misura dell'Informazione.

Trasmissione numerica in banda base: il canale di trasmissione, simboli e bit. Codifica PCM.

Trasmissione M-PAM interferenza tra simboli, effetto del rumore, probabilità di errore, il filtro adattato ed impulsi di Nyquist.

Trasmissione in banda traslata e con portanti in quadratura (QAM).



Recall: frequency domain

Signals as linear superposition (sum) of elementary sinusoid components

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{i2\pi f t} df$$

- $X(f)$ = Fourier transform of $x(t)$
- $X(f)$ is complex : Module and Phase
(or Real and Imaginary parts)

$$X(f) = F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-i2\pi f t} dt$$

Recall: Convolution

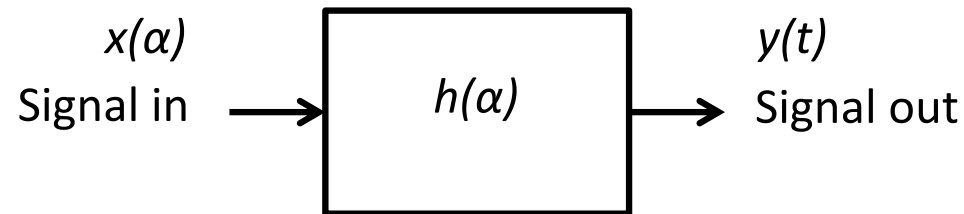
Constant-parameter linear filters (NO switches, NO time-variant components!!)
are characterized by

$H(f)$ transfer function in frequency domain

$$H(f) = F[h(t)]$$

$h(t)$ δ -response in time domain

$$h(t) = F^{-1}H(f)$$



The input $x(\alpha)$ can be described as a **linear superposition** (sum) of elementary **δ -pulses** of amplitude $x(\alpha)d\alpha$

THEREFORE

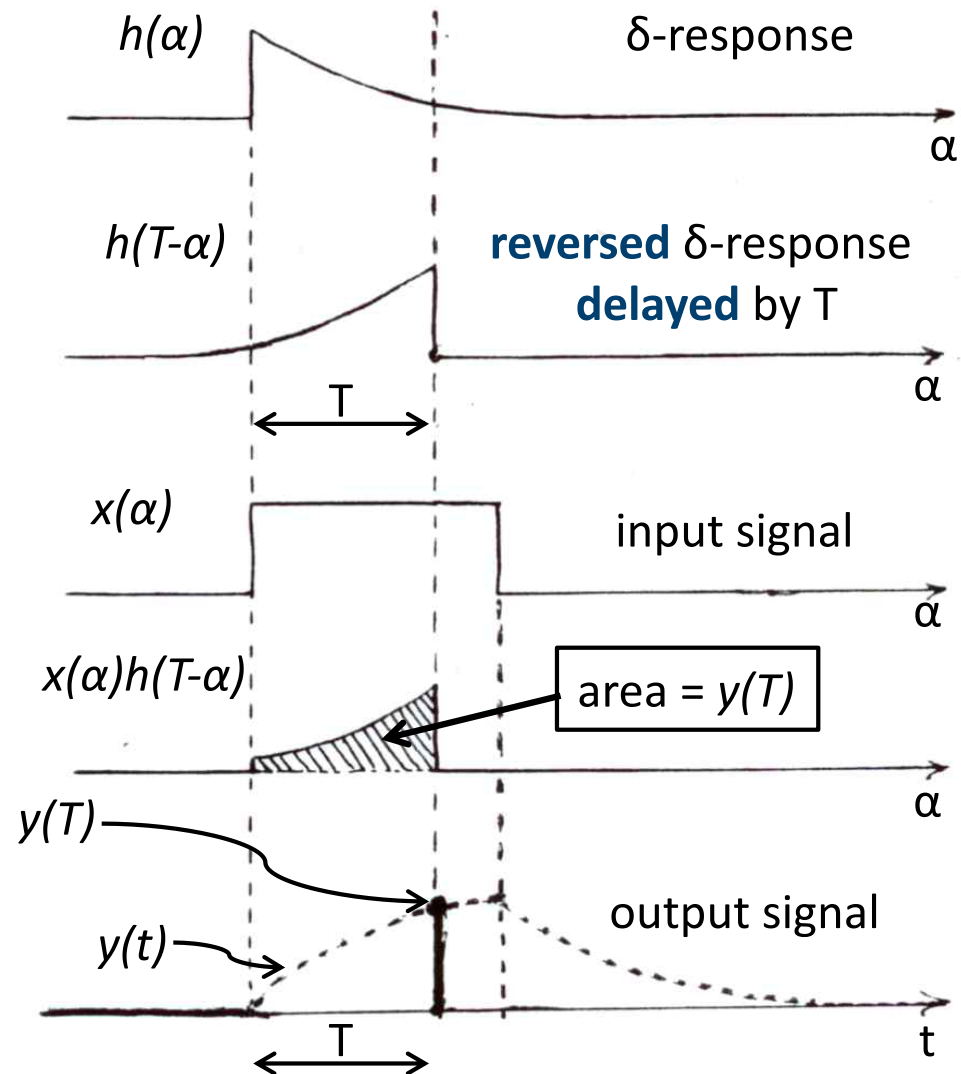
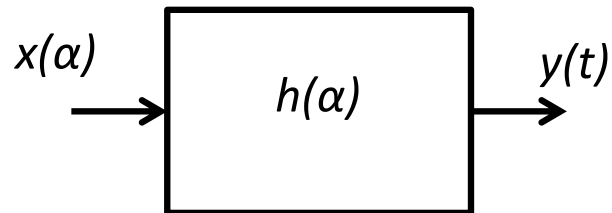
the output $y(t)$ can be described as a **linear superposition** (sum) of elementary **δ -pulse responses** $x(\alpha)d\alpha h(t-\alpha)$

$$y(t) = x(\alpha) * h(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(t - \alpha)d\alpha$$

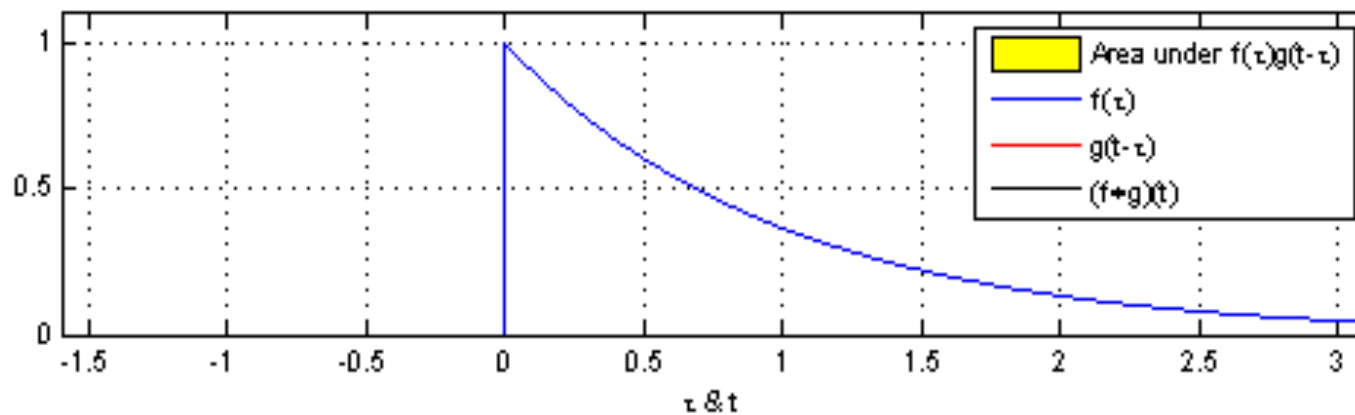
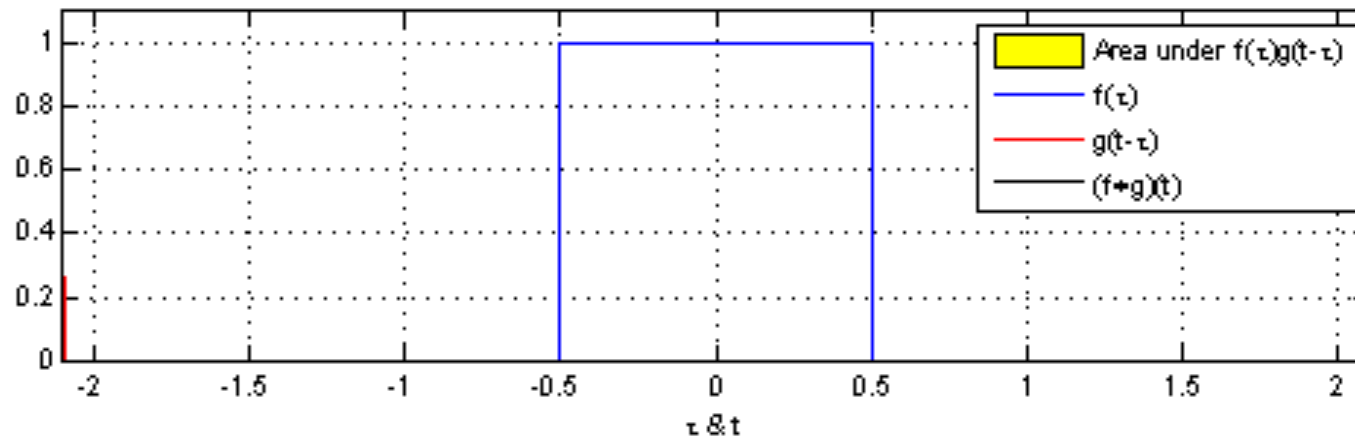


Recall: Computing the convolution

$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)h(t - \alpha)d\alpha$$



Recall: Computing the convolution



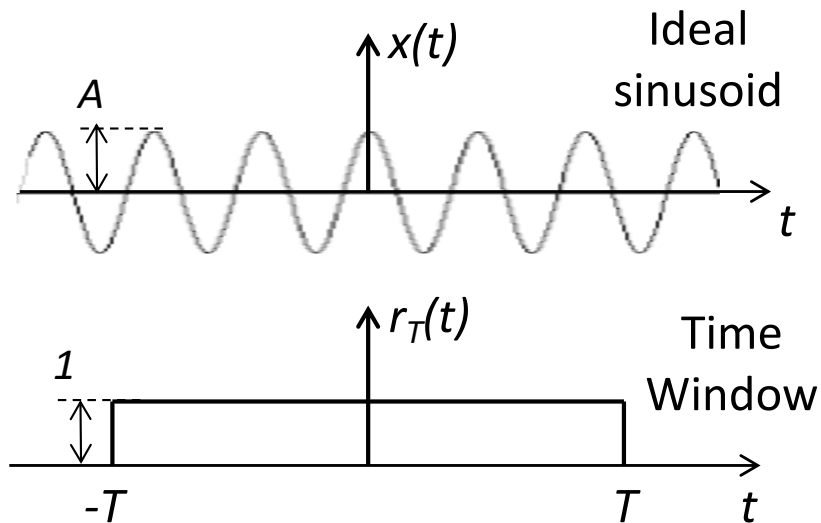
How does it change the convolution changing the exponential decay time?

Note on truncated signals

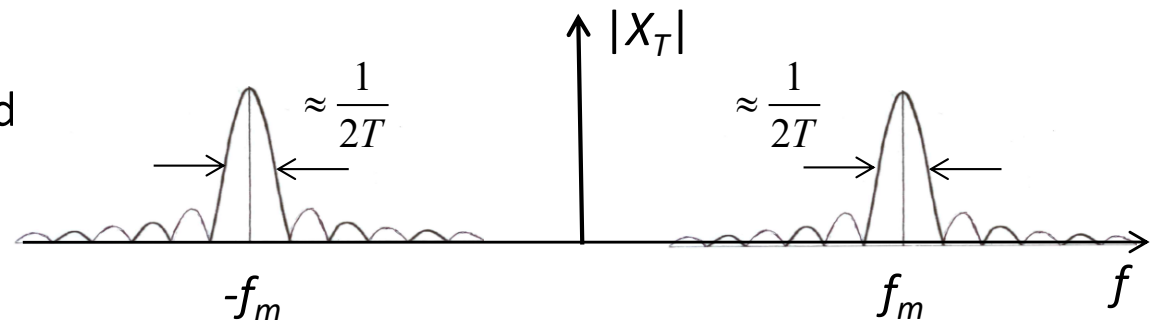
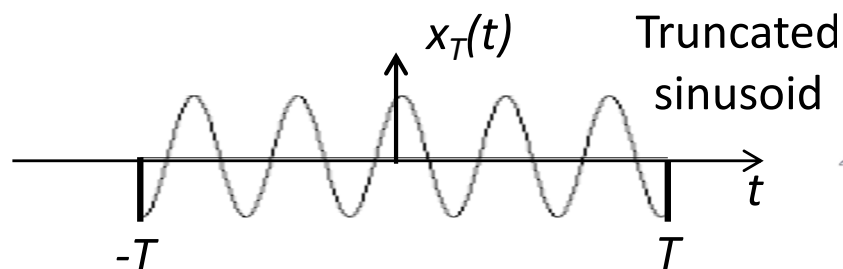
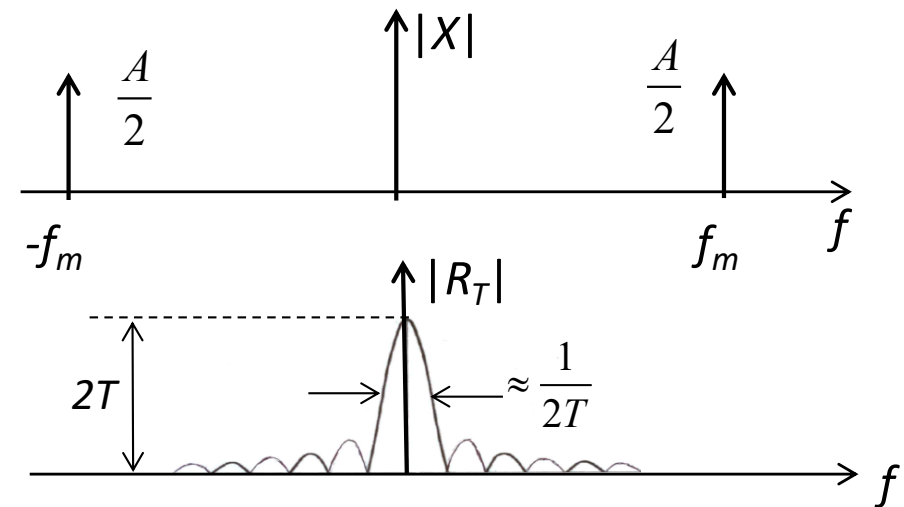
Note on truncated signals

Noteworthy case: truncated sinusoidal signal

seen in time domain



seen in frequency domain



$$x_T(t) = x(t) \cdot r_T(t)$$

$$\begin{array}{c} \xrightarrow{F} \\ \xleftarrow{F^{-1}} \end{array}$$

$$X_T(f) = X(f) * R_T(f)$$

Note on truncated signals

- In reality the signal is always available over a finite time interval: therefore, **in reality we always deal with truncated signals**
- cropping in time corresponds to **convolution** of the signal in the f domain with the transform of the rectangle (*sinc* function)
- the convolution spreads the signal in the f domain; that is, it makes it wider and smoother
- the narrower is the window $2T$, the wider is the *sinc* and more significant is the signal spreading in frequency
- Applying correctly the sampling theorem we see that: the **sampling frequency** f_s to be employed for a **truncated** sinusoid of frequency f_m is **NOT** $f_s \approx 2f_m$; it must be REMARKABLY HIGHER $f_s \gg 2f_m$

Energy signals and correlation functions

Signal Energy

The Energy E of a signal $x(t)$ is defined as

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T x^2(\alpha) d\alpha = \int_{-\infty}^{\infty} x^2(\alpha) d\alpha$$

Signals $x(t)$ with finite E are called **energy-signals**. Typical example: **pulse signals**

INTUITIVE VIEW OF ENERGY:

Let $x(t)$ be a voltage pulse on a unitary resistance $R=1 \Omega$,

Power= V^2/R then E is the energy dissipated in R by the pulse

A red rectangular stamp with the word "EXAMPLE" in bold, capital letters, tilted slightly to the right.

$$k_{xx}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T x(\alpha)x(\alpha + \tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)x(\alpha + \tau)d\alpha$$

$k_{xx}(\tau)$ gives the **degree of similarity** of $x(t)$ with itself **shifted by τ**

Energy = Autocorrelation at zero-shift

$$k_{xx}(0) = E$$



Signal Auto-Correlation Function (Energy-type)

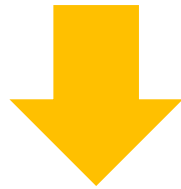
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EXAMPLE

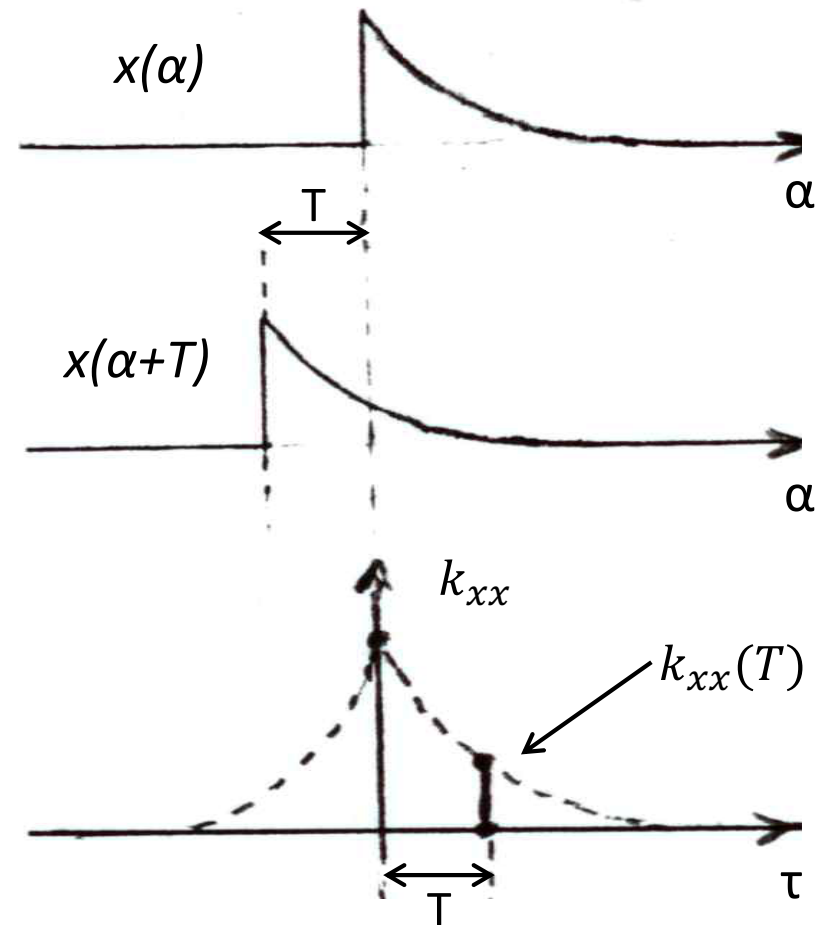
$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha + \tau)d\alpha$$

Case: exponential decay

$$x = 1(t)Ae^{-t/T_p}$$



$$k_{xx}(\tau) = A^2 \frac{T_p}{2} e^{-|\tau|/T_p}$$



Signal Auto-Correlation Function (Energy-type)

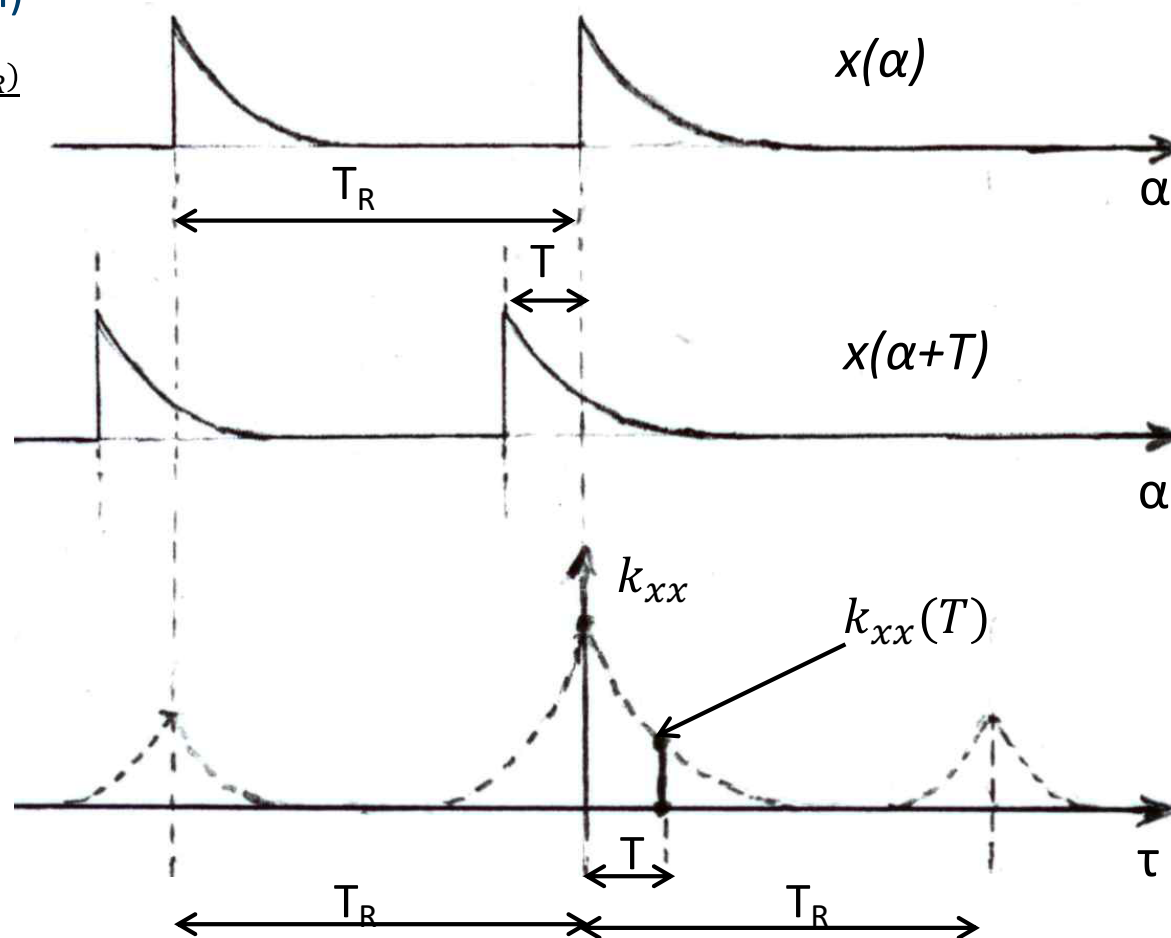
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$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha + \tau)d\alpha$$

EXAMPLE

Case: double pulse (exponential)

$$x = 1(t)Ae^{-\frac{t}{T_P}} + 1(t - T_R)Ae^{-\frac{(t-T_R)}{T_P}}$$



$$k_{xx}(\tau) = A^2 \frac{T_P}{2} e^{-|\tau|/T_P} + A^2 \frac{T_P}{2} e^{-|t-T_R|/T_P} + A^2 \frac{T_P}{2} e^{-|t+T_R|/T_P}$$

$$k_{xy}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T x(\alpha) y(\alpha + \tau) d\alpha = \int_{-\infty}^{\infty} x(\alpha) y(\alpha + \tau) d\alpha$$

- $x(t)$ and $y(t)$ are **two different** signals of energy-type
- $k_{xy}(\tau)$ gives the degree of similarity of $x(t)$ with $y(t)$ shifted by τ to left (towards earlier time)
- Various denominations for $k_{xy}(\tau)$:

Cross-Correlation function of x and y

Mutual-Correlation function of x and y

Cross-Correlation obtained by Convolution

Convolution

$$x * y = z(T)$$

is different from Crosscorrelation $k_{xy}(T)$

However

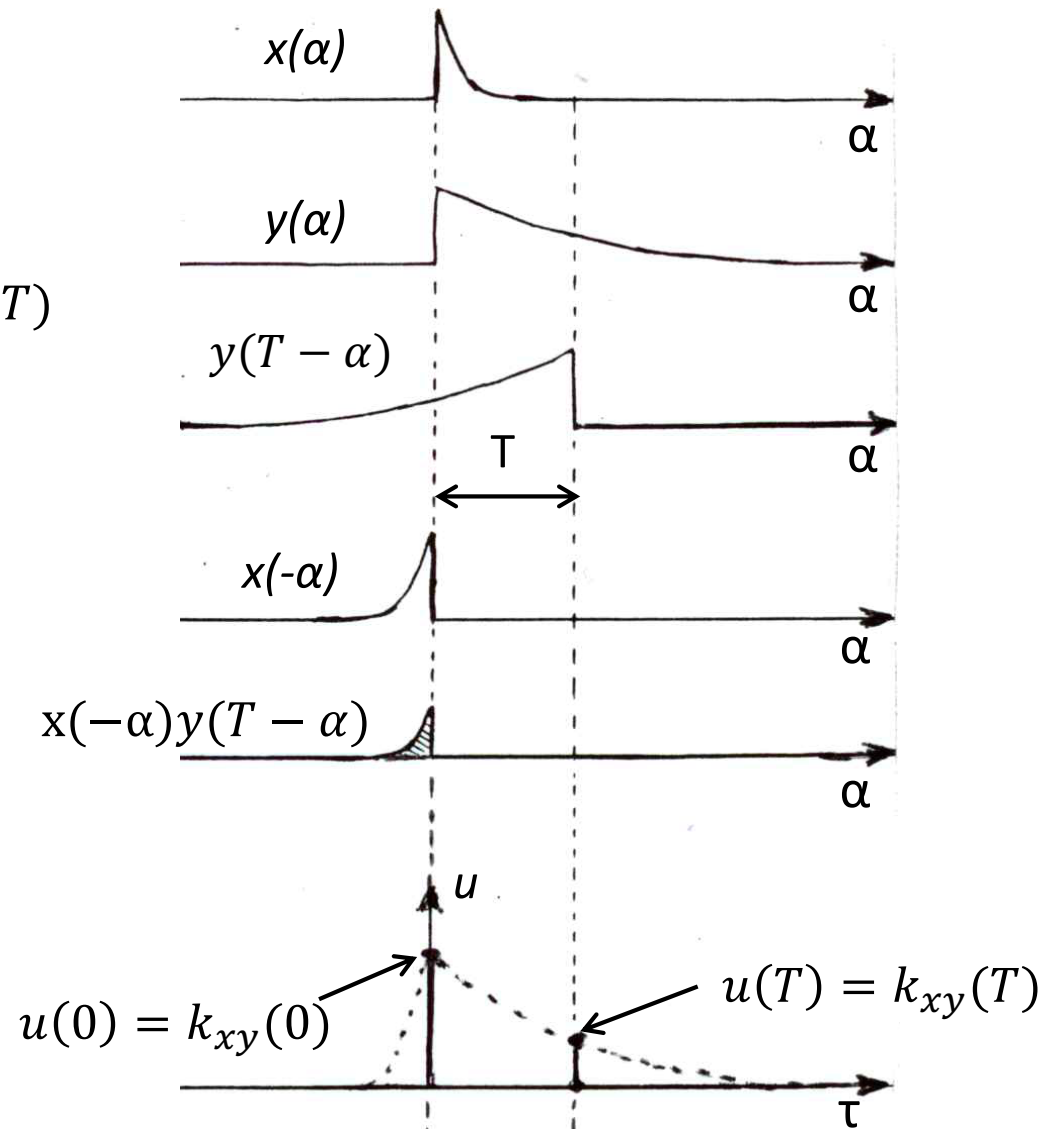
Convolution with **first term reversed**

$$x(-a) * y(a) = u(T)$$

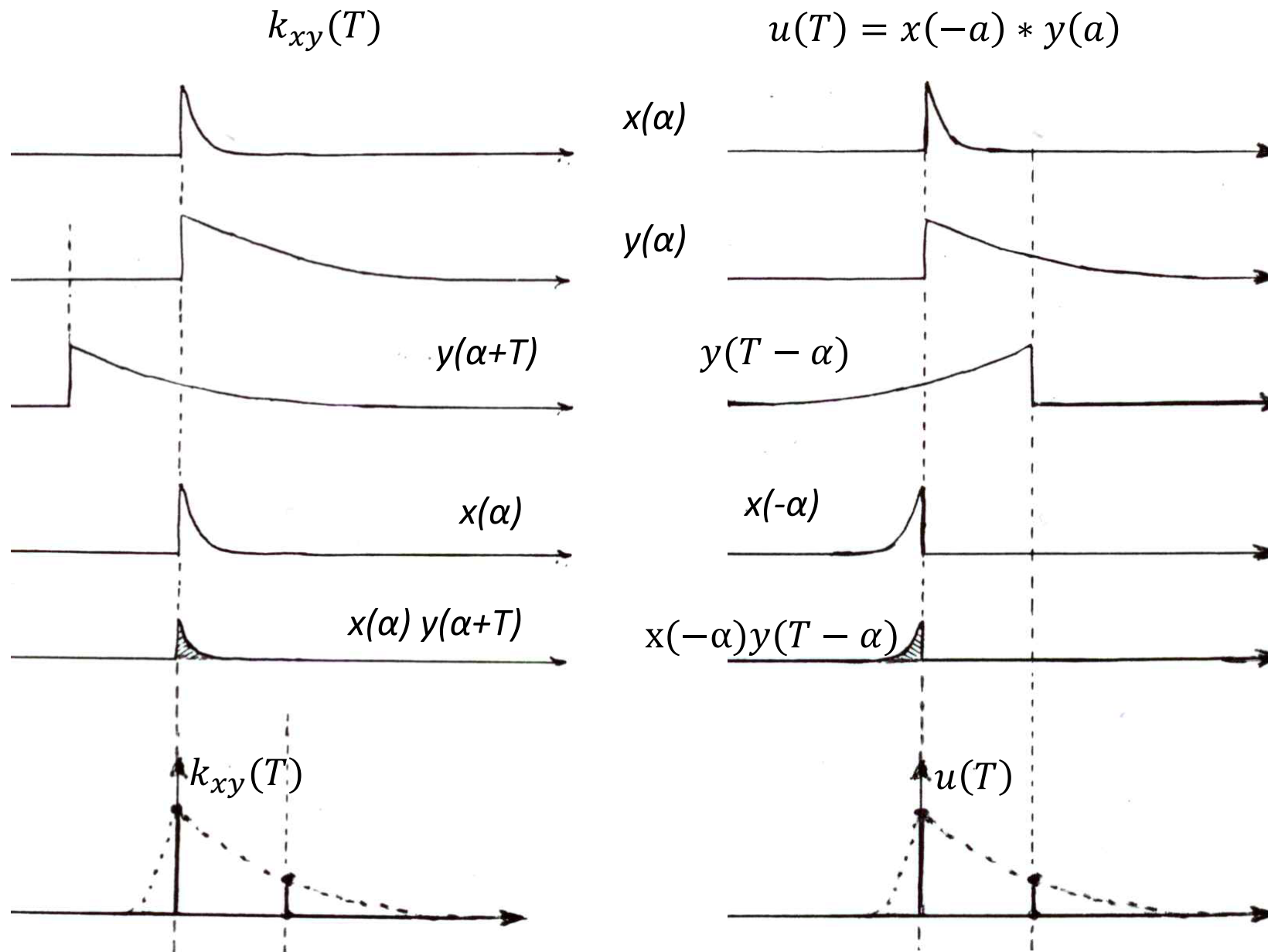
is equal to Crosscorrelation

$$u(T) = k_{xy}(T)$$

$$k_{xy}(T) = x(-a) * y(a)$$



Cross-Correlation obtained by Convolution



Energy Spectrum

Energy Spectrum

Energy signal $x(\alpha)$ with Fourier transform $X(f)$: by Parseval's theorem

$$E = \int_{-\infty}^{\infty} x^2(\alpha) d\alpha = \int_{-\infty}^{\infty} |X(f)|^2 df = 2 \int_0^{\infty} |X(f)|^2 df$$

$S_x(f) = |X(f)|^2$ is called the **Energy Spectrum** of the signal $x(\alpha)$

INTUITIVE VIEW OF ENERGY SPECTRUM:

- (1) Let $x(t)$ be voltage on a unitary resistance $R=1 \Omega$
- (2) $x(t)$ = sum of sinusoid components with frequency f and amplitude $|X(f)|df$
- (3) Sinusoids are orthogonal functions : **No power** from multiplication of **different components** (different f)

Every component (at frequency f) contributes an energy:

$$dE = 2 |X(f)|^2 df$$

EXAMPLE

Energy Spectrum

- Alternative definition of the **Energy Spectrum** is

$$S_x = F[k_{xx}]$$



- Knowing that $k_{xx} = x(-\alpha) * x(\alpha)$ we see that the two definitions are consistent

$$S_x = F[k_{xx}] = F[x(-\alpha) * x(\alpha)] = X(-f)X(f) = X^*(f)X(f) = |X(f)|^2$$

and by a basic property of Fourier transforms

$$E = k_{xx}(0) = \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

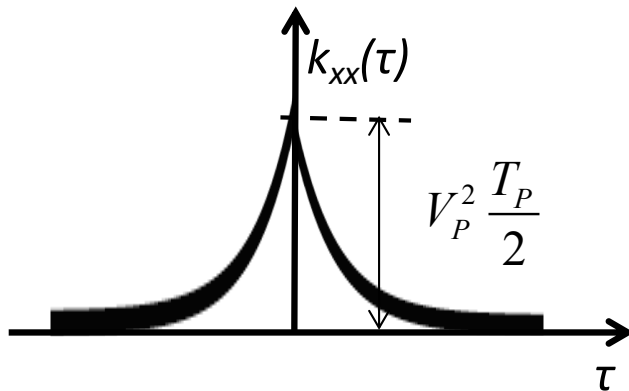


Example of Energy, Autocorrelation and Energy Spectrum

EXAMPLE



Exponential pulse: $x(t) = V_P e^{-t/T_P}$ $X(f) = V_P T_P \frac{1}{1 + j2\pi f T_P}$

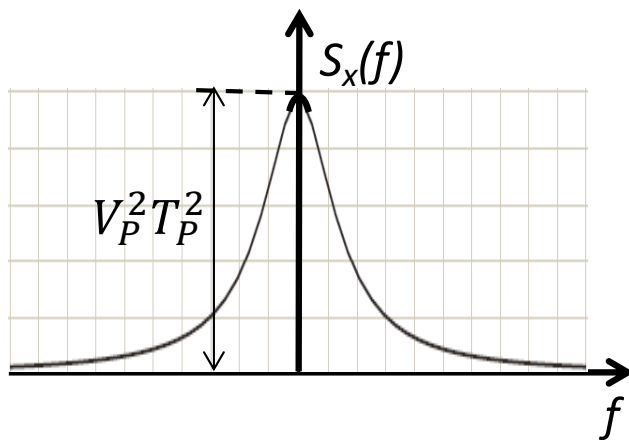


Autocorrelation function:

$$k_{xx}(\tau) = V_P^2 \frac{T_P}{2} e^{-|\tau|/T_P}$$

Energy:

$$E = k_{xx}(0) = V_P^2 \frac{T_P}{2}$$



Energy Spectrum:

$$S_x(f) = |X(f)|^2 = V_P^2 T_P^2 \frac{1}{1 + (2\pi f T_P)^2}$$

Energy:

$$E = \int_{-\infty}^{\infty} S_x(f) df = V_P^2 \frac{T_P}{2}$$

Power signals, Correlation Functions and Power Spectrum

For signals $x(t)$ that have NOT finite energy $E \rightarrow \infty$ (DC, sinusoids, periodic signals, etc.) the **Power P** is defined as the time-average

$$P = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x^2(\alpha)}{2T} d\alpha$$

Parseval theorem is valid for the entire integral $\int_{-\infty}^{+\infty}$

but NOT for the truncated integral $\int_{-T}^{+T}$

For computing P in f domain instead of truncated integral we use truncated signal $x_T(t)$

$$\begin{aligned} x_T(\alpha) &= x(\alpha) & \text{for } |\alpha| \leq T \\ x_T(\alpha) &= 0 & \text{for } |\alpha| > T \end{aligned}$$

We can thus exploit Parseval theorem: with $X_T(f) = F[x_T(\alpha)]$ we get

$$P = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{x_T^2(\alpha)}{2T} d\alpha = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{2T} df$$

The Power Spectrum of the signal $x(\alpha)$ is defined as the integrand

$$S_x(f) = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{2T} \quad \text{and} \quad P = \int_{-\infty}^{\infty} S_x(f) df$$

Just like power P , the autocorrelation of power signals is defined as **time-average**

$$K_{xx}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x(\alpha)x(\alpha + \tau)}{2T} d\alpha \quad \text{note that} \quad P = K_{xx}(0)$$

Also here we use truncated signal $x_T(\alpha)$ instead of truncated integral $\int_{-T}^T$

$$K_{xx}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x(\alpha)x(\alpha + \tau)}{2T} d\alpha = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)x_T(\alpha + \tau)}{2T} d\alpha$$

NB1: for finite T it is $\int_{-T}^T x(\alpha)x(\alpha + \tau)d\alpha \neq \int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha + \tau)d\alpha$
but for $\lim_{T \rightarrow \infty}$ the $=$ is valid

$$K_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{k_{xx,T}(\tau)}{2T}$$

An alternative definition of signal Power Spectrum is

$$S_x = F[K_{xx}(\tau)]$$



The two definitions are consistent

$$S_x(f) = F[K_{xx}(\tau)] = F\left[\lim_{T \rightarrow \infty} \frac{k_{xx,T}(\tau)}{2T}\right] = \lim_{T \rightarrow \infty} \frac{F[k_{xx,T}(\tau)]}{2T} = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{2T}$$

and

$$P = K_{xx}(0) = \int_{-\infty}^{\infty} S_x(f) df$$



$$K_{xy}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x(\alpha)y(\alpha + \tau)}{2T} d\alpha = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)y_T(\alpha + \tau)}{2T} d\alpha$$

$x(t)$ and $y(t)$ are two different signals, both power-type

$K_{xy}(\tau)$ measures the degree of similarity of $x(t)$ with $y(t)$ shifted by τ to left (towards earlier time)

If even only one of the two signals $x(t)$ and $y(t)$ is energy-type the energy type autocorrelation $k_{xy}(\tau)$ must be employed

(in fact, the power-type crosscorrelation vanishes $K_{xy}(\tau) = 0$ and the energy-type crosscorrelation $k_{xy}(\tau)$ is finite).

Energy-type (pulses etc.)

Energy $E = \int_{-\infty}^{\infty} x^2(\alpha) d\alpha$

Autocorrelation

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha + \tau) d\alpha$$

Energy spectrum

$$S_{x,e} = F[k_{xx}(\tau)] = |X(f)|^2$$

and

$$E = \int_{-\infty}^{\infty} S_{x,e}(f) df$$

Power-type (periodic waveforms etc.)

Power $P = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x^2(\alpha)}{2T} d\alpha$

Autocorrelation

$$K_{xx}(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T \frac{x(\alpha)x(\alpha + \tau)}{2T} d\alpha$$

Power spectrum

$$S_{x,p} = F[K_{xx}(\tau)] = \lim_{T \rightarrow \infty} \frac{|X_T(f)|^2}{2T}$$

and

$$P = \int_{-\infty}^{\infty} S_{x,p}(f) df$$