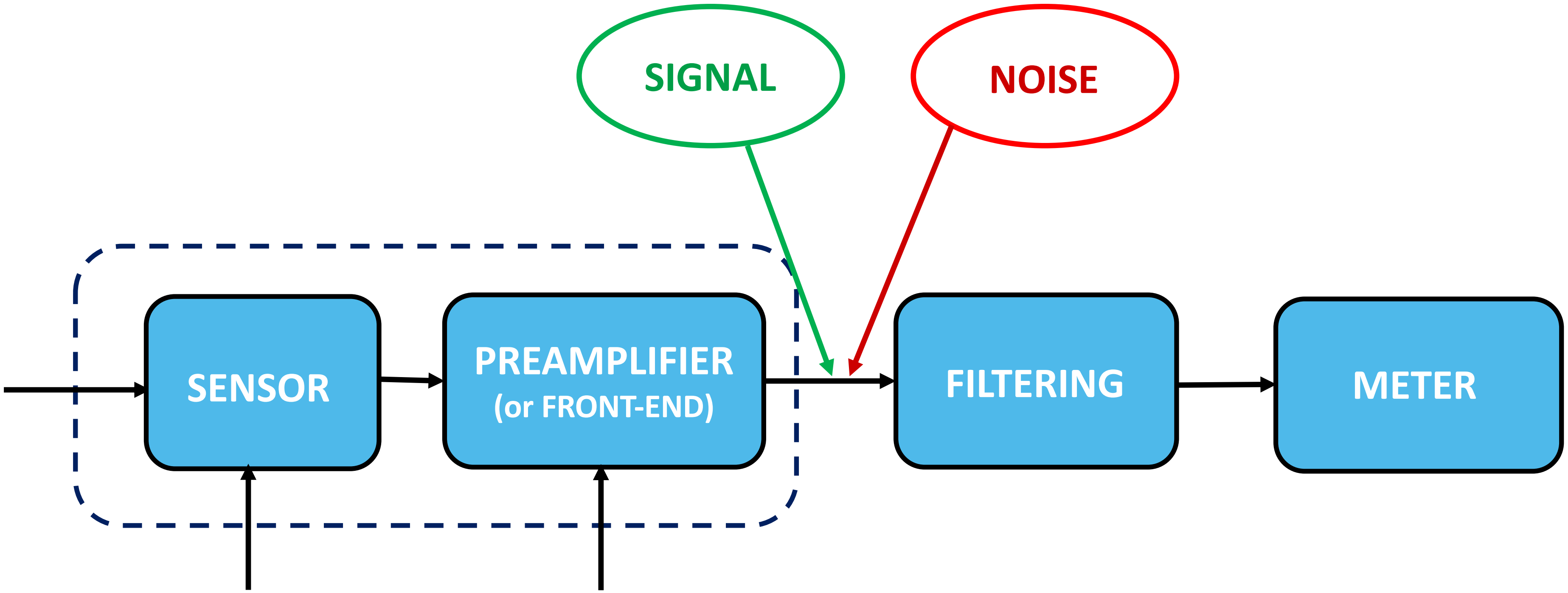
COURSE OUTLINE

- Introduction
- **Signals** and Noise
- Filtering
- Sensors and associated electronics

Set-Up for Sensor Measurements



SIGNIFICANT
Noise
of Sensor

SIGNIFICANT
Noise
of Preamp circuits
(or Front-end)

NEGLIGIBLE
Noise
of Filtering circuits
(*hopefully!*)

NEGLIGIBLE
Noise
of Meter circuits
(*hopefully!*)

Mathematical Description of Signals

- Time domain and frequency domain analysis
- Note on truncated signals: example of how we play with signals
- Energy signals and correlation functions
- Energy Spectrum
- Power signals, Correlation Functions and Power Spectrum

- *Book: Fourier transform and properties*
- *Book: Crosscorrelation and Convolution*

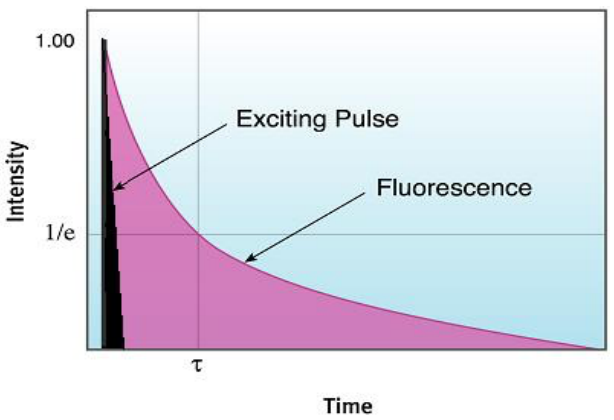
Time domain and frequency domain analysis of signals

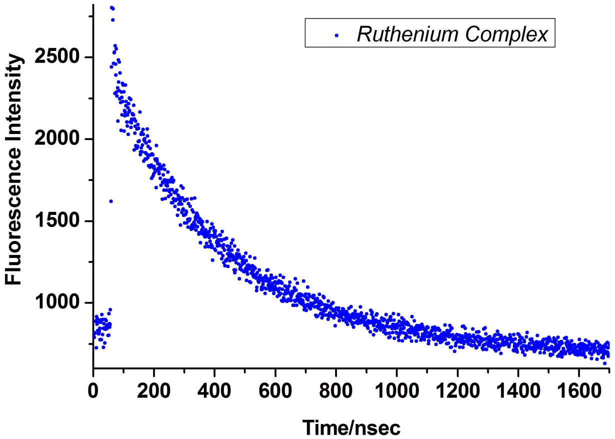
Signals: mathematical description

- **Signals** = electric variables *x* (voltage, current ...) that carry information
- In the domain of time *t* : **deterministic** functions *x = x(t)*

Example: exponential pulse

$$x = 1(t)e^{-t/T}$$





In the domain of frequency *f* (Fourier transform domain) can be considered linear superposition (sum) of elementary sinusoid components

FONDAMENTI DI SEGNALI E TRASMISSIONE

1. Segnali e sistemi continui
Segnali continui: scalino, impulso, esponenziali complessi, operazioni elementari sui segnali.
Sistemi Lineari Tempo-Invarianti: risposta impulsiva, convoluzione, correlazione.
Rappresentazione dei segnali nel dominio della frequenza: trasformata e serie di Fourier.
Densità spettrale di energia e potenza.
Dal tempo continuo al tempo discreto: teorema del campionamento, ricostruzione ed equivocazione nel tempo e nelle frequenze.
Trasformata di Fourier di Segnali discreti, energia, DFT.

2. Probabilità, processi casuali
Introduzione: definizioni, variabili casuali discrete e continue. Distribuzione e densità.
Probabilità condizionate: statistica indipendenza, regola di Bayes.
Prove ripetute (Bernoulli) e teoremi limite. Valori medi, quantili, momenti e correlazione.
Distribuzioni notevoli: normale, uniforme, binomiale, Poisson, esponenziale.
3. Processi casuali
Processi casuali: realizzazioni e medie d'insieme. Stazionarietà ed ergodicità.
Autocorrelazione e spettro di potenza. Predizione. Processi attraverso sistemi lineari.
Esempi e applicazioni: Rumore bianco e di quantizzazione.
Densità spettrale di processi modulati in fase e processi ciclostazionari. Stima spettrale non parametrica: periodogramma.
4. Informazione e trasmissione
Codifica di sorgente: quantizzazione, codifica di Huffman e misura dell'Informazione.
Trasmissione numerica in banda base: il canale di trasmissione, simboli e bit. Codifica PCM.
Trasmissione M-PAM interferenza tra simboli, effetto del rumore, probabilità di errore, il filtro adattato ed impulsi di Nyquist.
Trasmissione in banda traslata e con portanti in quadratura (QAM).

Recall: frequency domain

Signals as linear superposition (sum) of elementary sinusoid components

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{i2\pi f t}\, df$$

- *X(f)* = Fourier transform of x(t)
- *X(f)* is complex : Module and Phase

  (or Real and Imaginary parts)

$$X(f) = F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-i2\pi f t}\, dt$$

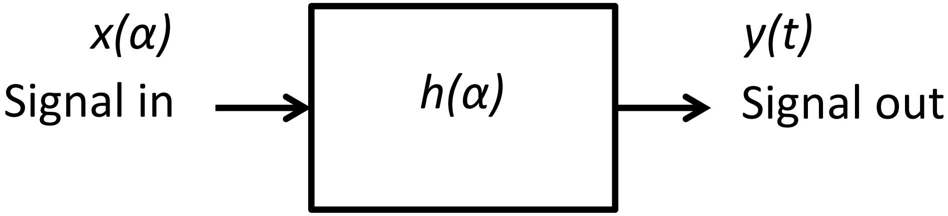
Recall: Convolution

Constant-parameter linear filters (NO switches, NO time-variant components!!)

are characterized by

H(f) transfer function in frequency domain $\qquad H(f) = F[h(t)]$

h(t) δ-response in time domain $\qquad h(t) = F^{-1}H(f)]$



The input x(α) can be described as a **linear superposition** (sum) of elementary **δ-pulses** of amplitude x(α)dα

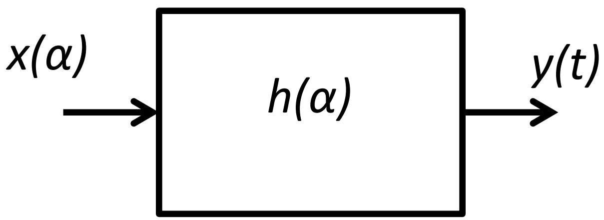
THEREFORE

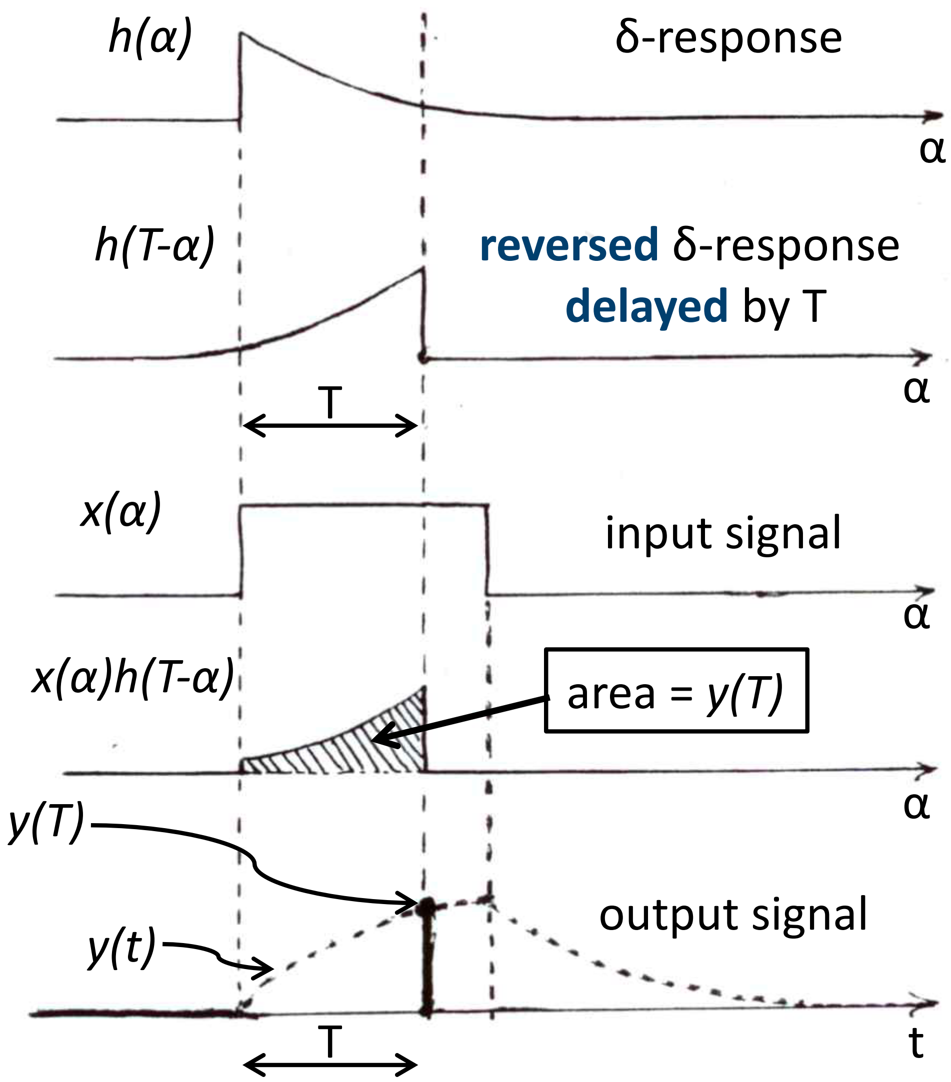
the output *y(t)* can be described as a **linear superposition** (sum) of elementary **δ-pulse responses** *x(α)dα h(t-α)*

$$\boxed{y(t) = x(\alpha) * h(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha}$$

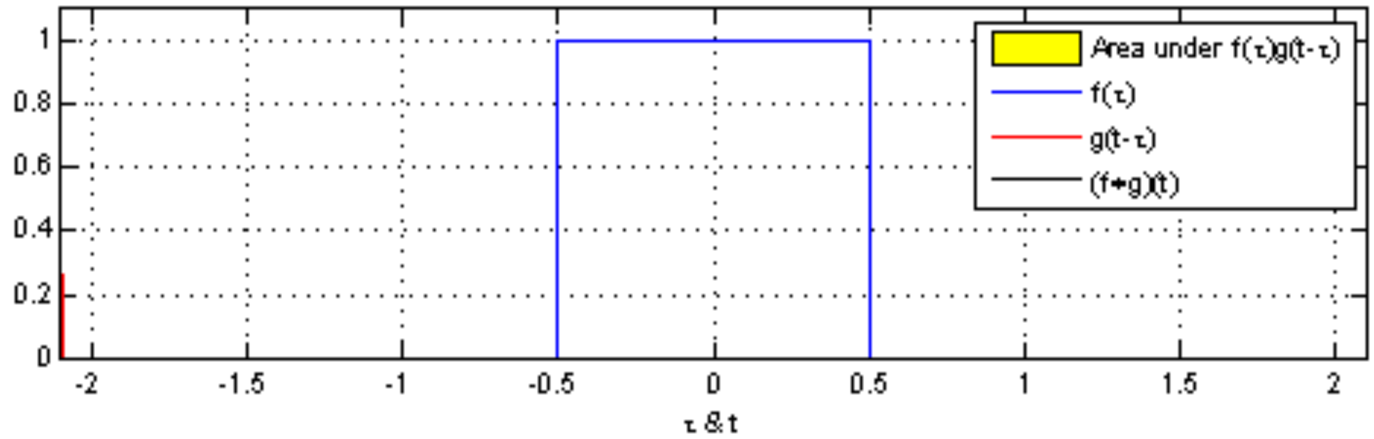
Recall: Computing the convolution

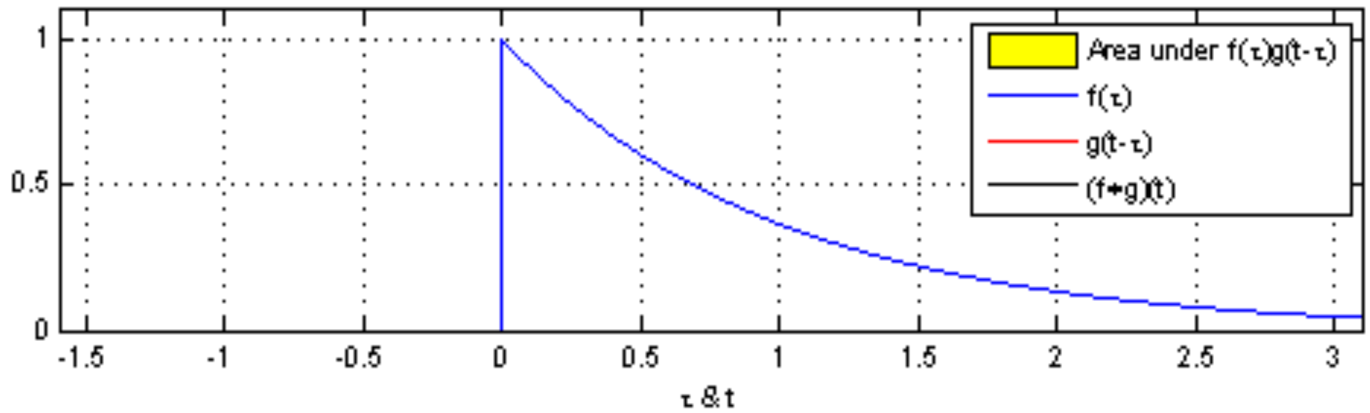
$$y(t) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha$$





Recall: Computing the convolution





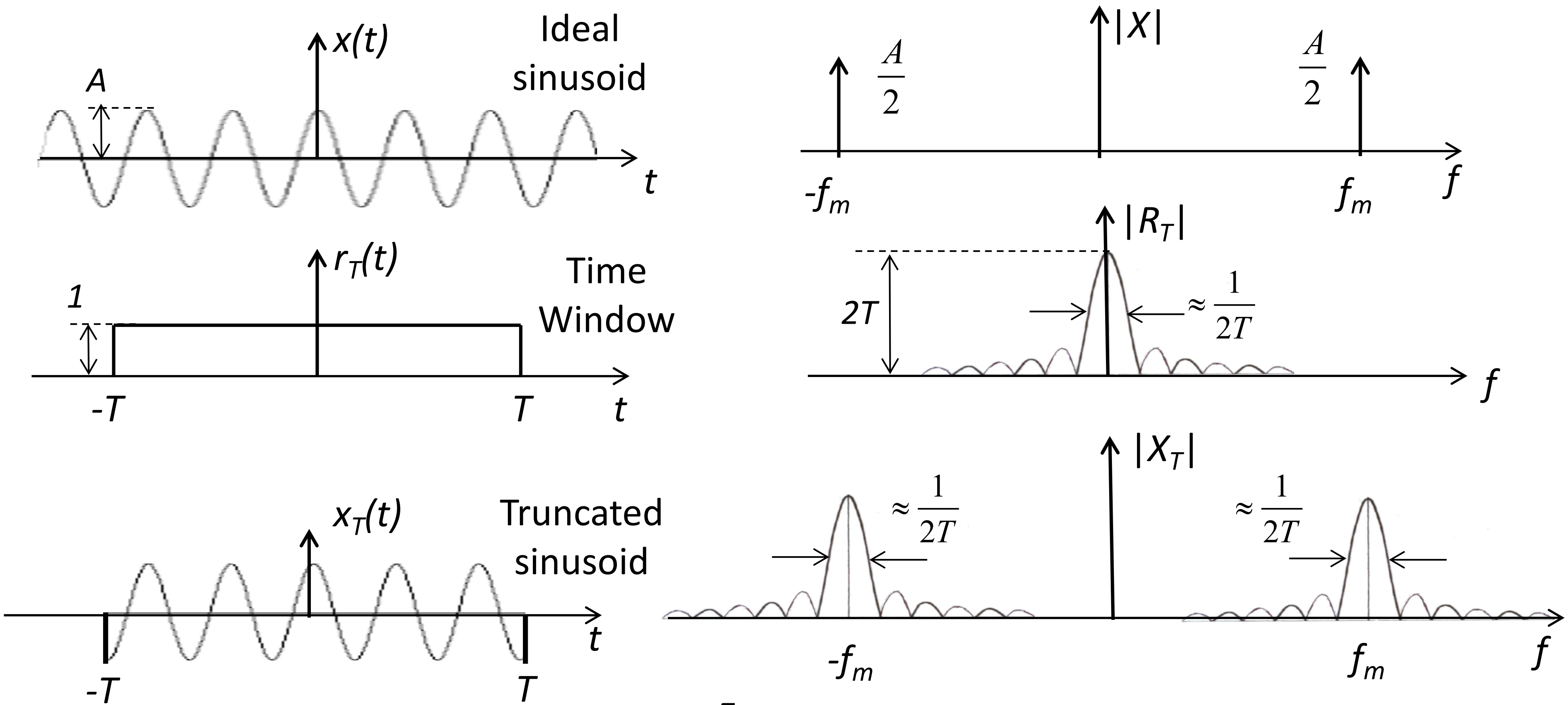
How does it change the convolution changing the exponential decay time?

Note on truncated signals

Note on truncated signals

Noteworthy case: truncated sinusoidal signal

seen in time domain | seen in frequency domain



$$x_T(t) = x(t) \cdot r_T(t) \quad \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \quad X_T(f) = X(f) * R_T(f)$$

Note on truncated signals

- In reality the signal is always available over a finite time interval: therefore, **in reality we always deal with truncated signals**
- cropping in time corresponds to **convolution** of the signal in the f domain with the transform of the rectangle (*sinc* function)
- the convolution spreads the signal in the f domain; that is, it makes it wider and smoother
- the narrower is the window $2T$, the wider is the *sinc* and more significant is the signal spreading in frequency
- Applying correctly the sampling theorem we see that: the **sampling frequency f_S** to be employed for a **truncated** sinusoid of frequency f_m is **NOT** $f_S \approx 2f_m$ **;** it must be REMARKABLY HIGHER $f_S >> 2f_m$

Energy signals and correlation functions

Signal Energy

The Energy *E* of a signal *x(t)* is defined as

$$E = \lim_{T\to\infty} \int_{-T}^{T} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$$

Signals *x(t)* with finite *E* are called **energy-signals**. Typical example: **pulse signals**

INTUITIVE VIEW OF ENERGY:

Let x(t) be a voltage pulse on a unitary resistance R=1 Ω,

Power=V^2/R then E is the energy dissipated in R by the pulse

Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

$k_{xx}(\tau)$ gives the **degree of similarity** of $x(t)$ with itself **shifted by τ**

Energy = Autocorrelation at zero-shift

$$k_{xx}(0) = E$$

Signal Auto-Correlation Function (Energy-type)

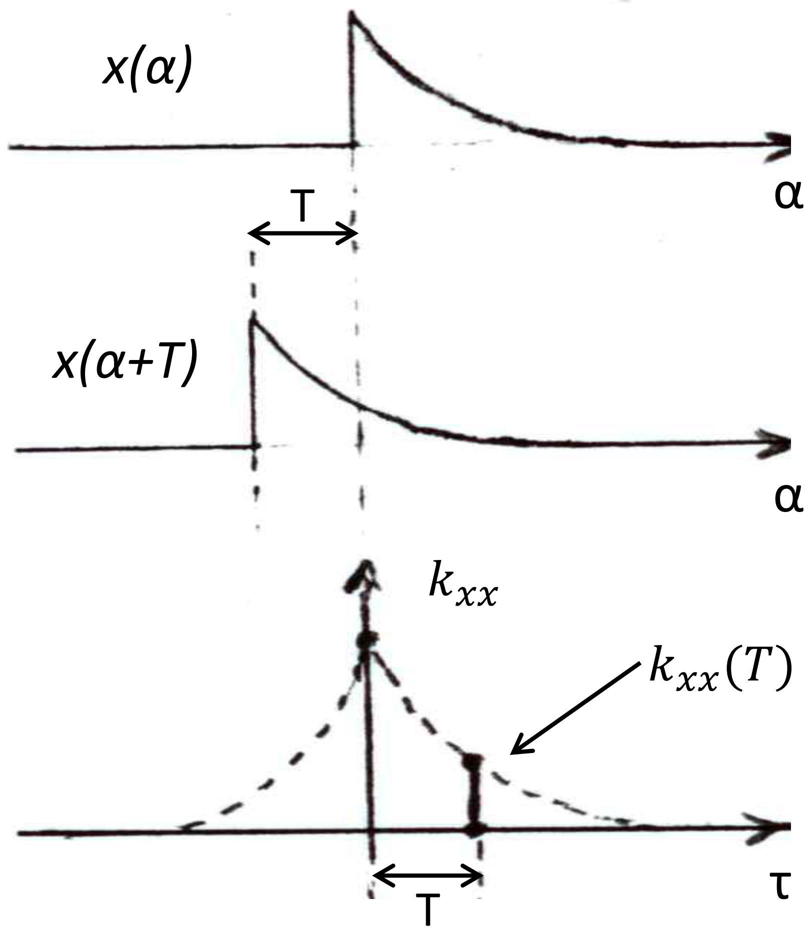
EXAMPLE

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Case: exponential decay

$$x = 1(t)Ae^{-t/T_P}$$

$$k_{xx}(\tau) = A^2 \frac{T_P}{2} e^{-|t|/T_P}$$

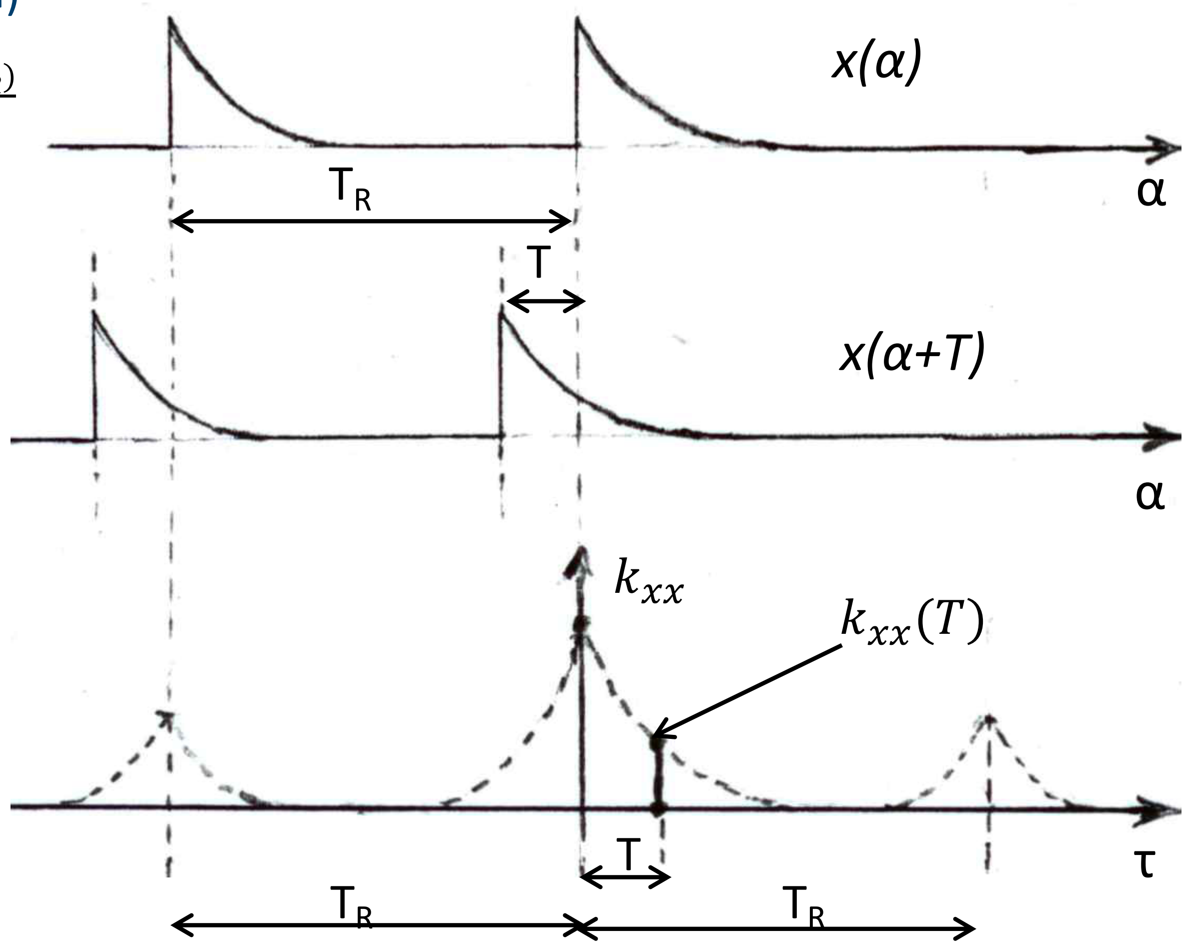


Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Case: double pulse (exponential)

$$x = 1(t)Ae^{-\frac{t}{T_P}} + 1(t-T_R)Ae^{-\frac{(t-T_R)}{T_P}}$$



$$k_{xx}(\tau) = A^2\frac{T_P}{2}e^{-|t|/T_P} + A^2\frac{T_P}{2}e^{-|t-TR|/T_P} + A^2\frac{T_P}{2}e^{-|t+TR|/T_P}$$

Signal Cross-Correlation Function (Energy-type)

$$k_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} x(\alpha)y(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)y(\alpha+\tau)d\alpha$$

- x(t) and y(t) are **two different** signals of energy-type
- ***$k_{xy}(\tau)$* gives the degree of similarity of *x(t)***

  with *y(t)* shifted by τ to left (towards earlier time)
- Various denominations for *$k_{xy}(\tau)$* :

  Cross-Correlation function of x and y

  Mutual-Correlation function of x and y

Cross-Correlation obtained by Convolution

Convolution

$$x * y = z(T)$$

is different from Crosscorrelation $k_{xy}(T)$

However

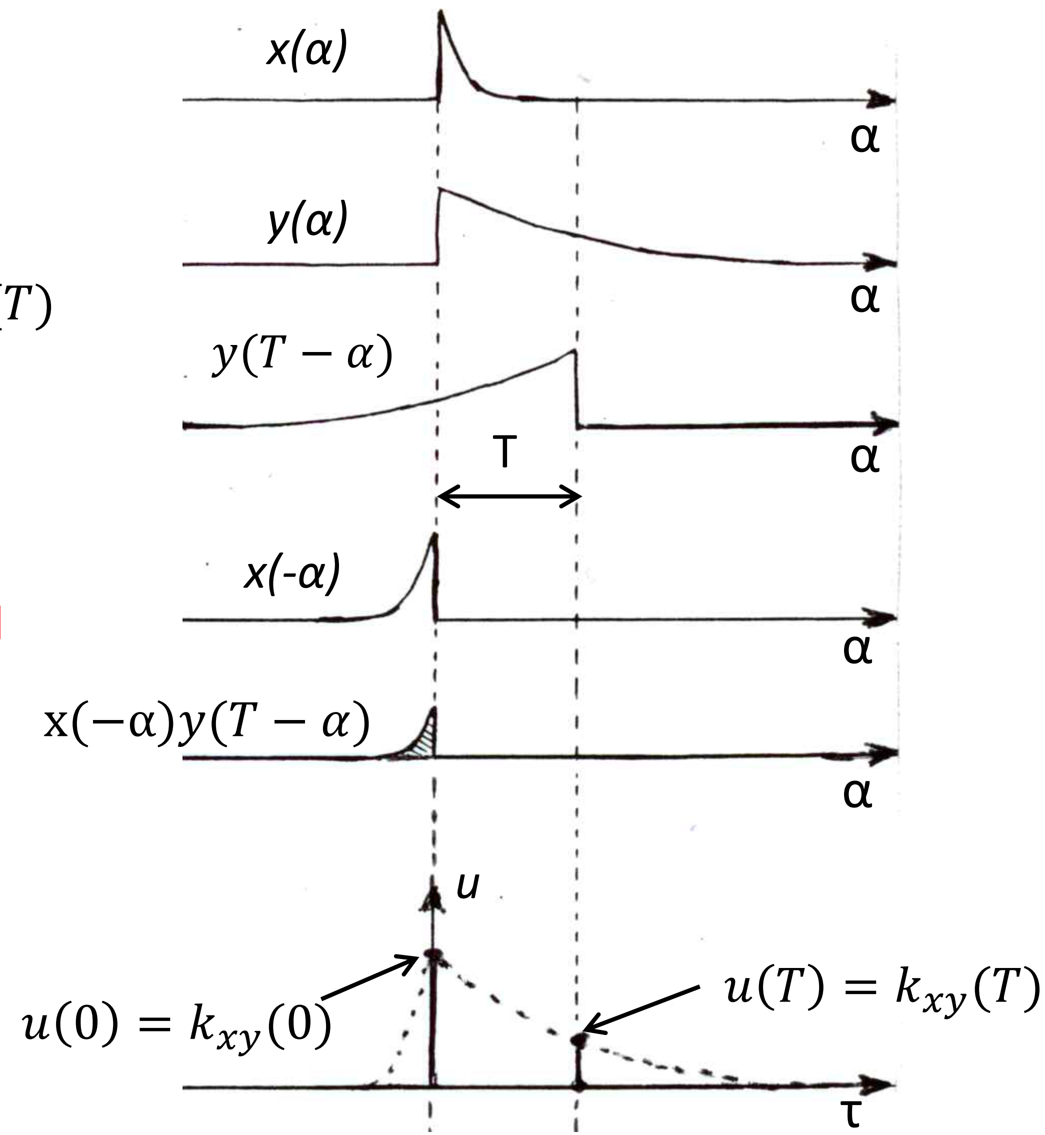
Convolution with **first term reversed**

$$x(-a) * y(a) = u(T)$$

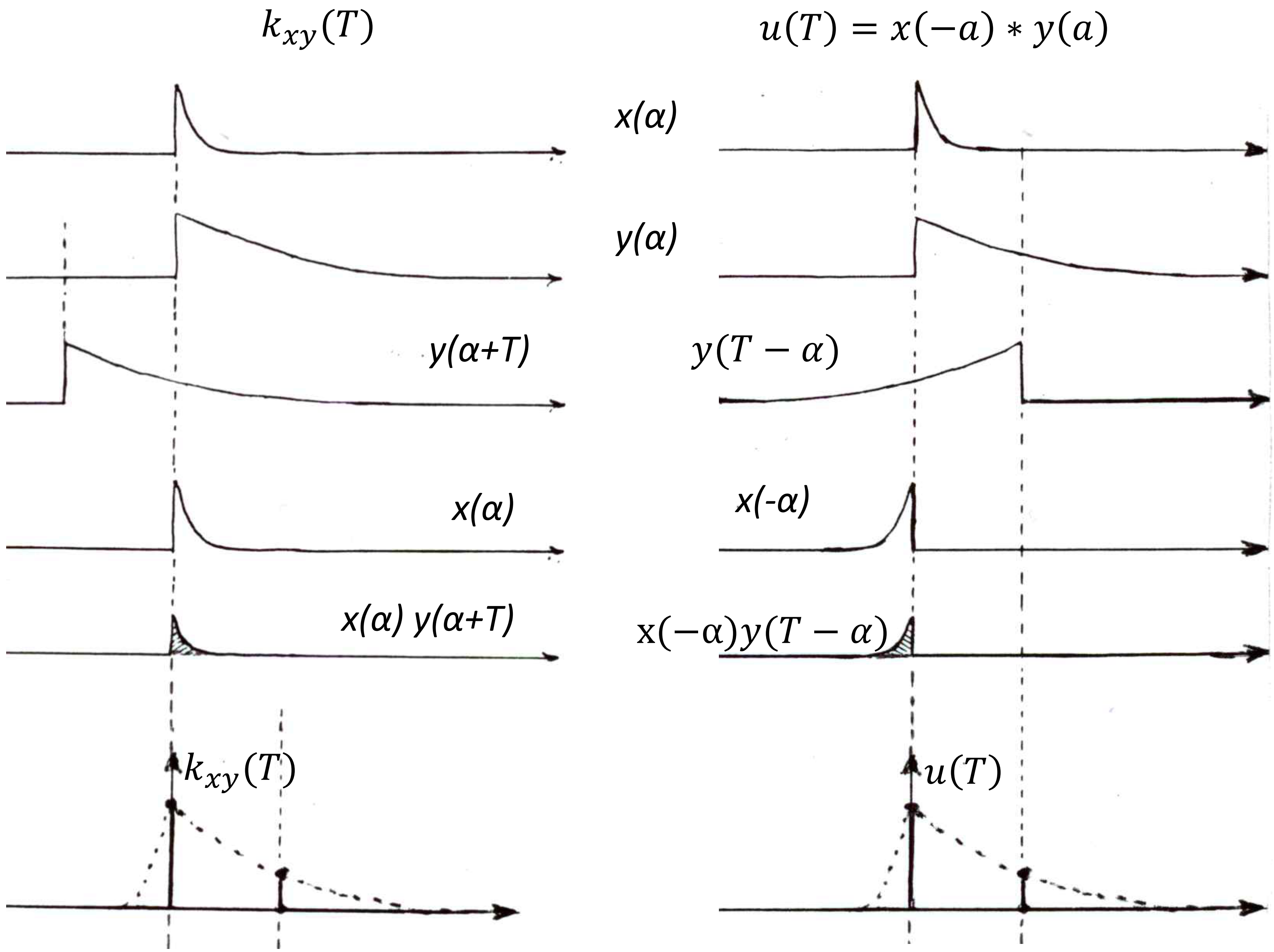
is equal to Crosscorrelation

$$u(T) = k_{xy}(T)$$

$$\boxed{k_{xy}(T) = x(-a) * y(a)}$$



Cross-Correlation obtained by Convolution



Energy Spectrum

Energy Spectrum

Energy signal *x(α)* with Fourier transform X(f): by Parseval's theorem

$$\boxed{E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty} |X(f)|^2 df = 2\int_{0}^{\infty} |X(f)|^2 df}$$

$S_x(f) = |X(f)|^2$ is called the **Energy Spectrum** of the signal *x(α)*

INTUITIVE VIEW OF ENERGY SPECTRUM:

(1) Let *x(t)* be voltage on a unitary resistance *R=1 Ω*

(2) *x(t)* = sum of sinusoid components with frequency f and amplitude $|X(f)|df$

(3) Sinusoids are orthogonal functions : **No power** from multiplication of **different components** (different f)

Every *component (at frequency f) contributes an energy:*

$$\boxed{dE = 2\,|X(f)|^2 df}$$

Energy Spectrum

- Alternative definition of the **Energy Spectrum** is

$$\boxed{S_x = F[k_{xx}]}$$

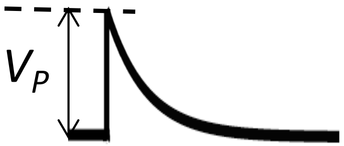
- Knowing that $k_{xx} = x(-\alpha) * x(\alpha)$ we see that the two definitions are consistent

$$S_x = F[k_{xx}] = F[x(-\alpha) * x(\alpha)] = X(-f)X(f) = X^*(f)X(f) = |X(f)|^2$$

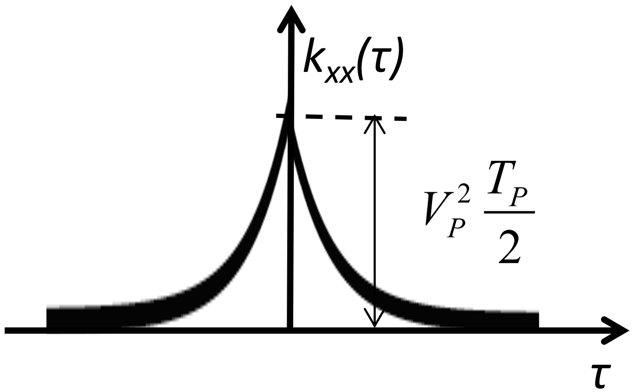
and by a basic property of Fourier transforms

$$\boxed{E = k_{xx}(0) = \int_{-\infty}^{\infty} S_x(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

Example of Energy, Autocorrelation and Energy Spectrum

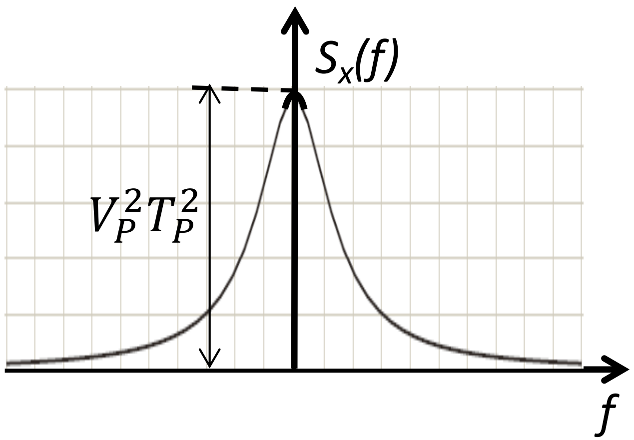


Exponential pulse: $x(t) = V_P e^{-t/T_P}$ $\quad X(f) = V_P T_P \frac{1}{1+j2\pi f T_P}$



Autocorrelation function: $k_{xx}(\tau) = V^2 \frac{T_P}{2} e^{-|t|/T_P}$

Energy: $E = k_{xx}(0) = V^2 \frac{T_P}{2}$



Energy Spectrum: $S_x(f) = |X(f)|^2 = {V^2}_P {T^2}_P \frac{1}{1+(2\pi f T_P)^2}$

Energy: $E = \int_{-\infty}^{\infty} S_x(f)df = V^2 \frac{T_P}{2}$

Power signals, Correlation Functions and Power Spectrum

Signal Power

For signals $x(t)$ that have NOT finite energy $E \rightarrow \infty$ (DC, sinusoids, periodic signals, etc.) the **Power P** is defined as the time-average

$$P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$$

Parseval theorem is valid for the entire integral $\int_{-\infty}^{+\infty}$

but NOT for the truncated integral $\int_{-T}^{+T}$

For computing P in f domain instead of truncated integral we use truncated signal $x_T(t)$

$$x_T(\alpha) = x(\alpha) \qquad \text{for } |\alpha| \leq T$$
$$x_T(\alpha) = 0 \qquad \text{for } |\alpha| > T$$

We can thus exploit Parseval theorem: with $X_T(f) = F[x_T(\alpha)]$ we get

$$P = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{{x_T}^2(\alpha)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} df$$

The Power Spectrum of the signal $x(\alpha)$ is defined as the integrand

$$S_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} \qquad \text{and} \qquad P = \int_{-\infty}^{\infty} S_x(f) df$$

Signal Auto-Correlation Function (Power-type)

Just like power P, the autocorrelation of power signals is defined as **time-average**

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha \qquad \text{note that} \qquad P = K_{xx}(0)$$

Also here we use truncated signal *$x_T(\alpha)$* instead of truncated integral $\int_{-T}^{T}$

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)x_T(\alpha+\tau)}{2T} d\alpha$$

NB1: for finite T it is $\int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha \neq \int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha$

but for $\lim_{T\to\infty}$ the = is valid

Signal Auto-Correlation Function and Power Spectrum

$$\boxed{K_{xx}(\tau) = \lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}}$$

An alternative definition of signal <u>Power Spectrum</u> is

$$\boxed{S_x = F[K_{xx}(\tau)]}$$

The two definitions are consistent

$$S_x(f) = F[K_{xx}(\tau)] = F[\lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}] = \lim_{T\to\infty} \frac{F[k_{xx,T}(\tau)]}{2T} = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

and

$$\boxed{P = K_{xx}(0) = \int_{-\infty}^{\infty} S_x(f)df}$$

Signal Cross-Correlation Function (Power-type)

$$K_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)y(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)y_T(\alpha+\tau)}{2T} d\alpha$$

x(t) and y(t) are two different signals, both power-type

***$K_{xy}(\tau)$* measures the degree of similarity of *x(t)* with *y(t)* shifted by τ to left (towards earlier time)**

If even only one of the two signals x(t) and y(t) is energy-type the energy type autocorrelation $k_{xy}(\tau)$ must be employed

(in fact, the power-type crosscorrelation vanishes $K_{xy}(\tau) = 0$ and the energy-type crosscorrelation $k_{xy}(\tau)$ is finite).

Energy-signals and power-signals compared

Energy-type (pulses etc.)

Energy $E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$

Autocorrelation

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Energy spectrum

$$S_{x,e} = F[k_{xx}(\tau)] = |X(f)|^2$$

and

$$E = \int_{-\infty}^{\infty} S_{x,e}(f)df$$

Power-type (periodic waveforms etc.)

Power $P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$

Autocorrelation

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha$$

Power spectrum

$$S_{x,p} = F[K_{xx}(\tau)] = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

and

$$P = \int_{-\infty}^{\infty} S_{x,p}(f)df$$